Improper Integrals

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By an *improper integral* we will mean an integral of the form

$$\int_a^\infty f(x)\,dx. \tag{1}$$

The goal of this note is to carefully define, and then study the properties of, improper integrals. To this end, let $a \in \mathbb{R}$ and let f be a function that is Riemann integrable on every finite subinterval of $[a, \infty)$. We then define

$$\int_a^\infty f(x)\,dx = \lim_{A\to\infty} \int_a^A f(x)dx$$

provided the limit on the right-hand side exists. In this case we say that the improper integral (1) *converges* and that its value is that of the limit. Otherwise we say (1) *diverges*. It turns out that the theory of improper integrals closely mirrors that of infinite series, and we will prove several results which can be viewed as direct analogues of results about series. For example, one can use the usual limit laws to immediately verify that improper integrals (when convergent) obey the usual linearity of definite integrals. Another more powerful result is the following alternate formulation of convergence.

Theorem 1 (Cauchy Criterion)**.** *The improper integral* (1) *converges if and only if for every* $\epsilon > 0$ *there is an* $M \geq a$ *so that for all* $A, B \geq M$ *we have*

$$\left| \int_A^B f(x)\,dx \right| < \epsilon.$$

Proof. Suppose that the improper integral converges to L. Let $\epsilon > 0$. Using the definition of convergence, choose $M \geq a$ so large that if $A \geq M$ then

$$\left| \int_a^A f(x)\,dx - L \right| < \frac{\epsilon}{2}.$$

Then if $B \geq M$ as well we have

$$\begin{aligned}
\left| \int_A^B f(x)\,dx \right| &= \left| \int_a^B f(x)\,dx - \int_a^A f(x)dx \right| \\
&= \left| \int_a^B f(x)\,dx - L + L - \int_a^A f(x)dx \right| \\
&\leq \left| \int_a^B f(x)\,dx - L \right| + \left| L - \int_a^A f(x)dx \right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

That's the easy implication. For the converse, now suppose the stated Cauchy criterion holds. For natural numbers $n \geq a$ let

$$a_n = \int_a^n f(x)\, dx.$$

Let $\epsilon > 0$ and choose $M \geq a$ as stated in the hypothesis. Then if $m, n \geq M$ we have

$$|a_n - a_m| = \left| \int_m^n f(x)\, dx \right| < \epsilon,$$

which shows that $\{a_n\}$ is a Cauchy sequence. Let its limit be L. Again let $\epsilon > 0$ and this time choose $M \geq a$ so that $|a_n - L| < \epsilon/2$ and

$$\left| \int_A^B f(x)\, dx \right| < \frac{\epsilon}{2}$$

whenever $n, A, B \geq M$. If $A \geq M + 1$ then $[A] \geq M$ so that

$$\begin{aligned} \left| \int_a^A f(x)\, dx - L \right| &= \left| \int_a^{[A]} f(x)\, dx - L \int_{[A]}^A f(x)\, dx \right| \\ &\leq |a_{[A]} - L| + \left| \int_{[A]}^A f(x)\, dx \right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This proves that the improper integral in question converges to L.

□

We say that the improper integral (1) *converges absolutely* if the improper integral

$$\int_a^\infty |f(x)|\, dx$$

converges. Perhaps not surprisingly, absolute convergence of an improper integral bears the same relationship to ordinary convergence as in the case of infinite series. Specifically we have the next result.

Theorem 2 (Absolute convergence implies convergence.). *If the improper integral* (1) *converges absolutely then it converges.*

Proof. We make use of the Cauchy criterion. Let $\epsilon > 0$. Since the improper integral of $|f(x)|$ converges we can find an $M \geq a$ so that for all $A, B \geq M$ we have

$$\left| \int_A^B |f(x)|\, dx \right| < \epsilon.$$

But the integral of $|f(x)|$ is nonnegative, so we have

$$\left| \int_A^B f(x)\, dx \right| \leq \int_A^B |f(x)|\, dx < \epsilon$$

which proves that the improper integral of $f(x)$ satisfies the Cauchy criterion, and hence converges. □

As with infinite series, it is often the case that one can prove convergence of an improper integral even though one cannot give a closed form expression for its limit. An example is the integral

$$\int_1^\infty \frac{t-[t]}{t^2}\,dt$$

which is simply *defined* to be the constant $\gamma = 0.5772....$ We would therefore like to have a way of testing for the convergence of an improper integral without having to evaluate it. Here are some common tests.

Theorem 3 (Comparison Test)**.** *Suppose that f and g are Riemann integrable on every finite subinterval of $[a,\infty)$ and that $0 \le f(x) \le g(x)$ for all $x \ge a$. If $\int_a^\infty g(x)\,dx$ converges, so does $\int_a^\infty f(x)\,dx$.*

Proof. For $A \ge a$ Let

$$F(A) = \int_a^A f(x)\,dx$$

and

$$G(A) = \int_a^A g(x)\,dx.$$

Our hypotheses imply that $F(A) \le G(A)$ and that both of these functions are increasing. Furthermore, $G(A)$ tends to a limit L as $A \to \infty$. It follows that

$$F(A) \le G(A) \le L.$$

Since $F(A)$ is increasing and bounded above by L, it must also converge to a limit as $A \to \infty$, which is what we needed to prove. $\square$

We frequently need a means to apply this test even though the improper integrals in question do not have the same lower limit. This can be achieved by making use of the next lemma.

Lemma 1. *If f is Riemann integrable on every finite subinterval of $[a,\infty)$ and $b \ge a$, then $\int_a^\infty f(x)\,dx$ converges if and only if $\int_b^\infty f(x)\,dx$ converges.*

Proof. If $M \ge b$ then the integral appearing in the Cauchy criterion for both improper integrals is the same. The result follows at once. $\square$

We often will find it convenient to make comparisons with improper integrals of the form

$$\int_1^\infty \frac{1}{x^p}\,dx$$

where $p > 0$ is a fixed real number. If $p \ne 1$ we have

$$\int_1^A \frac{1}{x^p}\,dx = \left.\frac{x^{1-p}}{1-p}\right|_1^A = \frac{A^{1-p}}{1-p} + \frac{1}{p-1}.$$

If $p < 1$ then this diverges as $A \to \infty$, whereas if $p > 1$ this tends to the limit $1/(p-1)$ as $A \to \infty$. If $p = 1$ then we instead have

$$\int_1^A \frac{1}{x^p}\,dx = \ln A$$

which diverges as $A \to \infty$. We can summarize these computations as

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & \text{if } p > 1, \\ \infty & \text{if } 0 < p \leq 1. \end{cases}$$

Although the comparison test can be quite useful, there are times when directly comparing the integrands of two improper integrals is inconvenient. In this situation one can often appeal to the following result.

Theorem 4 (Limit Comparison Test)**.** *Suppose that f and g are nonnegative and Riemann integrable on every finite subinterval of $[a, \infty)$, and that*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

exists. If $\int_a^\infty g(x)\,dx$ converges then so does $\int_a^\infty f(x)\,dx$. If $L \neq 0$ then the converse also holds.

Proof. Choosing $\epsilon = 1$ in the definition of the limit, we find that there is an $M \geq a$ so that for all $x \geq M$ we have

$$\frac{f(x)}{g(x)} < L + 1.$$

Because the functions in question are nonnegative this gives $0 \leq f(x) \leq (L+1)g(x)$. If we assume that $\int_a^\infty g(x)\,dx$ converges, then so, too, does $\int_M^\infty (L+1)g(x)\,dx$ by linearity and the lemma, which according to the comparison test tells us that $\int_M^\infty f(x)\,dx$ converges as well. Another appeal to the lemma gives the result. The final statement of the theorem follows from the first since

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \frac{1}{L}$$

in this case. $\square$

Although the hypotheses of the Comparison and Limit Comparison Tests only apply to nonnegative integrands, we can apply them more generally by taking the absolute value of an integrand prior to using the test. In both cases the tests are then for absolute convergence. To be a bit more specific, given an arbitrary integrand f, we instead consider $|f|$ and compare it to a nonnegative integrand g whose improper integral converges, thereby demonstrating the absolute convergence of the improper integral of f.