Homework #7 Solutions

#1 Let $I = \langle f(x), g(x)\rangle$. Since $F[x]$ is a PID there is an $h(x) \in F[x]$ so that $I = \langle h(x)\rangle$. But this implies that $h(x)$ divides both $f(x)$ and $g(x)$. As $f(x)$ and $g(x)$ are relatively prime this can only happen if $h(x)$ is a unit in $F[x]$. Hence $\langle f(x), g(x)\rangle = I = \langle h(x)\rangle = F[x]$. Since $1 \in F[x]$ we conclude that there exist $r(x), s(x) \in F[x]$ so that $r(x)f(x) + s(x)g(x) = 1$.

#2 According to Exercise 1, there are polynomials $r_1(x), s_1(x) \in F[x]$ so that $r(x)f(x) + s(x)g(x) = 1$. Multiplying both sides of this equation by $c(x)$ we obtain $r_2(x)f(x) + s_2(x)g(x) = c(x)$, where $r_2(x) = c(x)r_1(x)$ and $s_2(x) = c(x)s_1(x)$. Now apply the division algorithm to obtain

$$\begin{aligned} r_2(x) &= a(x)g(x) + r(x) \\ s_2(x) &= b(x)f(x) + s(x) \end{aligned}$$

where $a(x), b(x), r(x), s(x) \in F[x]$, $r(x) = 0$ or $\deg r(x) < \deg g(x)$, and $s(x) = 0$ or $\deg s(x) < \deg f(x)$. Now substitute these expressions into $r_2(x)f(x) + s_2(x)g(x) = c(x)$ and rearrange:

$$c(x) = f(x)g(x)(a(x) + b(x)) + r(x)f(x) + s(x)g(x).$$

If $f(x)g(x)(a(x) + b(x)) \neq 0$ then it has degree greater than or equal to $\deg f(x)g(x)$. But then $c(x) - r(x)f(x) - s(x)g(x)$ is nonzero as well and has degree strictly less that $\deg f(x)g(x)$ because of the degree restrictions on $c(x), r(x), s(x)$. But then we have

$$\deg(f(x)g(x)) \leq \deg(f(x)g(x)(a(x)+b(x))) = \deg(c(x)-r(x)f(x)-s(x)g(x)) < \deg(f(x)g(x))$$

which is impossible. We conclude, therefore, that $f(x)g(x)(a(x) + b(x)) = 0$ and that $c(x) = r(x)f(x) + s(x)g(x)$. Dividing by $f(x)g(x)$ we obtain

$$\frac{c(x)}{f(x)g(x)} = \frac{r(x)}{g(x)} + \frac{s(x)}{f(x)}$$

with the required degree restrictions on $r(x)$ and $s(x)$.

#3 This is substantially easier than Exercise 2. Use the division algorithm to write $c(x) = q(x)f(x) + r(x)$ with $q(x), r(x) \in F[x]$ where $r(x) = 0$ or $\deg r(x) < \deg f(x)$. Since $\deg c(x) < \deg f(x)^2$, it follows that $\deg(q(x)f(x)) = \deg(c(x) - r(x)) < \deg f(x)^2$. That is, $\deg q(x) + \deg f(x) < 2 \deg f(x)$, which implies that $\deg q(x) < \deg f(x)$. So, dividing both sides of $c(x) = q(x)f(x) + r(x)$ by $f(x)$ we obtain

$$\frac{c(x)}{f(x)^2} = \frac{q(x)}{f(x)^2} + \frac{r(x)}{f(x)}$$

with the desired degree restrictions on $q(x)$ and $r(x)$.