Modern Algebra II
Spring 2007

Final Exam Practice

Exercise 1. Find the minimal polynomial of $\sqrt{3+\sqrt{3}}$ over $\mathbb{Q}$.

Exercise 2. Let F be a field, $f(x) \in F[x]$ a polynomial of degree 4 and E a splitting field for $f(x)$ over F. Prove that $[E : F] \leq 4! = 24$.

Exercise 3. Let $n, p \in \mathbb{Z}^+$ with p prime.

a. If $g(x) \in \mathbb{Z}_p[x]$ is irreducible of degree dividing n, prove that $g(x)$ divides $x^{p^n} - x$.

b. Prove that $x^{p^n} - x$ is equal to the product of all of the monic irreducible polynomials in $\mathbb{Z}_p[x]$ whose degree divides n.

Exercise 4. If α and β are complex numbers that are transcendental over $\mathbb{Q}$, prove that at least one of $\alpha\beta$ and $\alpha + \beta$ is also transcendental over $\mathbb{Q}$. [*Hint:* Consider the polynomial $(x - \alpha)(x - \beta)$ and argue by contradiction.]

Exercise 5. Let $p \in \mathbb{Z}^+$ be a prime. Let E/F be a field extension of degree p. If a is an element of E *not* in F, prove that the minimal polynomial of a over F has degree p.

Exercise 6. Let $p \in \mathbb{Z}^+$ be a prime, $F = \mathrm{GF}(p)$ and $f(x) \in F[x]$ be irreducible. If a is a root of $f(x)$ in some extension of F, prove that $F(a)$ is a splitting field for $f(x)$.

Exercise 7. Let F be a field of characteristic 0 and let $f(x) \in F[x]$ be irreducible. Prove that $f(x)$ cannot have multiple zeros.

Exercise 8. Prove that $\pi^2 - 1$ is algebraic over $\mathbb{Q}(\pi^3)$.

Exercise 9. Recall that an idempotent in a ring R with unity is an element $a \in R$ so that $a^2 = a$. Let p be an odd prime and let $k \in \mathbb{Z}^+$. Show that the only idempotents in $\mathbb{Z}_{p^k}$ are $a = 0, 1$.

Exercise 10. Let R be a ring and let I, J be nonzero ideals in R. If $I \cap J = \{0\}$, prove that the nonzero elements of I and J are zero divisors in R. [*Hint:* What happens if you multiply an element in I by an element in J?]

Exercise 11. Let F be a field. Define what it means for a set V to be a *vector space* over F. Define the terms *linearly dependent, linearly independent* and *basis.*