3.3.2 We have
\[
\frac{1}{z(z + 1)} = \frac{1}{z} \frac{1}{z + 1} = \frac{1}{z^2} \frac{1}{1 + 1/z} = \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n}
\]
since \(|z| > 1\) implies that \(|1/z| < 1\). Multiplying the \(1/z^2\) into the series and reindexing we have
\[
\frac{1}{z(z + 1)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{z^n}
\]
for \(|z| > 1\). Since Laurent series are unique, this must be the desired expansion.

3.3.4 We first note that we have the partial fraction expansion
\[
\frac{1}{z(z - 1)(z - 2)} = \frac{1}{z} \left( \frac{-1}{z - 1} + \frac{1}{z - 2} \right).
\]
(a) For \(0 < |z| < 1\) we have
\[
\frac{-1}{z - 1} = \frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n
\]
and
\[
\frac{1}{z - 2} = \frac{-1}{2} \frac{1}{1 - z/2} = \frac{-1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^n}
\]
Hence, in this region we have
\[
\frac{1}{z(z - 1)(z - 2)} = \frac{1}{z} \left( \sum_{n=0}^{\infty} z^n - \frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^n} \right) = \frac{1}{z} \sum_{n=0}^{\infty} \left( 1 - \frac{1}{2^{n+1}} \right) z^n.
\]
Multiplying the \(1/z\) through the sum and reindexing we have
\[
\frac{1}{z(z - 1)(z - 2)} = \sum_{n=0}^{\infty} \left( 1 - \frac{1}{2^{n+2}} \right) z^n + \frac{1}{2z}
\]
for \(0 < |z| < 1\). Uniqueness of Laurent series guarantees this is the desired expansion.

(b) When \(1 < |z| < 2\) we have
\[
\frac{-1}{z - 1} = \frac{-1}{z} \frac{1}{1 - 1/z} = \frac{-1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} = -\sum_{n=1}^{\infty} \frac{1}{z^n}
\]
and, as above,

\[ \frac{1}{z - 2} = -\frac{1}{2} \frac{1}{1 - z/2} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^n}. \]

Thus

\[ \frac{1}{z(z - 1)(z - 2)} = \frac{1}{z} \left( -\sum_{n=1}^{\infty} \frac{1}{z^n} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^n} \right) = -\sum_{n=0}^{\infty} \frac{z^{n-1}}{2^{n+1}} - \sum_{n=1}^{\infty} \frac{1}{z^{n+1}}. \]

Reindexing we find

\[ \frac{1}{z(z - 1)(z - 2)} = -\sum_{n=0}^{\infty} \frac{z^{n+2}}{2^{n+2}} - \frac{1}{2z} - \sum_{n=2}^{\infty} \frac{1}{z^n} \]

and once again the uniqueness of Laurent series tells us that this is the expression we sought.

3.3.8 If \( f \) and \( g \) are both analytic with zeros of order \( k \) at \( z_0 \) then we can write \( f(z) = (z - z_0)^k \phi(z) \) and \( g(z) = (z - z_0)^k \psi(z) \) where \( \phi \) and \( \psi \) are analytic wherever \( f \) and \( g \) are (in particular, in some neighborhood of \( z_0 \)) and \( \phi(z_0) \neq 0, \psi(z_0) \neq 0 \). It follows that for \( z \neq z_0 \) we have

\[ \frac{f(z)}{g(z)} = \frac{\phi(z)}{\psi(z)}. \]

Since \( \psi(z_0) \neq 0 \) and both \( \psi \) and \( \phi \) are continuous at \( z_0 \) we have

\[ \lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{\phi(z)}{\psi(z)} = \frac{\phi(z_0)}{\psi(z_0)}. \]

This proves that \( f/g \) has a removable singularity at \( z_0 \). The problem is then finished by appealing to the following result.

**Proposition.** Let \( f \) be analytic at \( z_0 \) with a zero of order \( k \) there. Write \( f(z) = (z - z_0)^k \phi(z) \). Then \( \phi(z) \) is analytic at \( z_0 \) and \( \phi(z_0) = f^{(k)}(z_0)/k! \).

**Proof.** We already know that \( \phi(z) \) is analytic at \( z_0 \). We can therefore write

\[ \phi(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \]

in some neighborhood of \( z_0 \). Then

\[ f(z) = (z - z_0)^k \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=k}^{\infty} a_{n-k} (z - z_0)^n \]
in some neighborhood of \( z_0 \). Applying the uniqueness of Taylor series to this expression we find that

\[
a_0 = \frac{f^{(k)}(z_0)}{k!}.\]

On the other hand, from the original expression defining the \( a_n \) we know that \( a_0 = \phi(z_0) \). The result follows.

\[\square\]

3.3.18 The function \( e^{1/z} \) is analytic on \( \mathbb{C} \setminus \{0\} \) and therefore has an isolated singularity at \( z_0 = 0 \). Appealing to the Taylor series for \( e^{1/z} \) we find that for \( z \neq 0 \) we have

\[
e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \sum_{n=1}^{\infty} \frac{1/n!}{z^n}.
\]

By uniqueness, this must be the Laurent series expansion of \( e^{1/z} \) on \( \mathbb{C} \setminus \{0\} \). However, we know that the Laurent series coefficients are given by

\[
a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{e^{1/z}}{z^{n+1}} \, dz
\]

and

\[
b_n = \frac{1}{2\pi i} \int_{\gamma} e^{1/z} z^{n-1} \, dz.
\]

Comparing to the series expression above we find that we must have

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{e^{1/z}}{z^{n+1}} \, dz = 0
\]

for \( n \geq 1 \),

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{e^{1/z}}{z} \, dz = 1
\]

and

\[
\frac{1}{2\pi i} \int_{\gamma} e^{1/z} z^{n-1} \, dz = \frac{1}{n!}
\]

for \( n \geq 1 \). Hence

\[
\int_{\gamma} z^n e^{1/z} \, dz = \begin{cases} 0 & \text{if } n \leq -2 \\ \frac{2\pi i}{(n+1)!} & \text{if } n \geq -1. \end{cases}
\]

3.R.4 Since \( e^z \) is entire, for any \( z \in \mathbb{C} \) we have

\[
f(z) = e^{z^2} = \sum_{n=0}^{\infty} \frac{(z^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}.
\]
Uniqueness of Taylor series guarantees that the expression on the right is the Taylor series for $e^{z^2}$ at the origin. In particular, this means that the coefficient of $z^k$ appearing on the right hand side must be given by $f^{(k)}(0)/k!$. Hence

$$\frac{f^{(68)}(0)}{68!} = \frac{1}{34!}$$

or $f^{(68)}(0) = 68!/34!$.

3.R.12 Since $f(z)$ is analytic for $|z| < 1$, we know that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

for all $|z| < 1$. Let $z \in \mathbb{C}$. Then, according to our hypothesis we have

$$\left| \frac{f^{(n)}(0)}{n!} z^n \right| < \frac{M^n}{n!} |z|^n = \frac{(M|z|)^n}{n!}$$

for every $n \geq 0$. The series

$$\sum_{n=0}^{\infty} \frac{(M|z|)^n}{n!}$$

converges to $e^{M|z|}$. It follows that the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

is absolutely convergent. Since $z \in \mathbb{C}$ was arbitrary, this means that the radius of convergence of the latter series must be infinite and hence that series represents an entire function. Since $f$ agrees with this series for $|z| < 1$, we find that the series provides and extension of $f$ to an entire function.