3.3.2 We have

$$\frac{1}{z(z+1)} = \frac{1}{z}\frac{1}{z+1} = \frac{1}{z^2}\frac{1}{1+1/z} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n}$$

since $|z| > 1$ implies that $|1/z| < 1$. Multiplying the $1/z^2$ into the series and reindexing we have

$$\frac{1}{z(z+1)} = \sum_{n=2}^{\infty}\frac{(-1)^n}{z^n}$$

for $|z| > 1$. Since Laurent series are unique, this must be the desired expansion.

3.3.4 We first note that we have the partial fraction expansion

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{z}\left(\frac{-1}{z-1} + \frac{1}{z-2}\right).$$

(a) For $0 < |z| < 1$ we have

$$\frac{-1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

and

$$\frac{1}{z-2} = \frac{-1}{2}\frac{1}{1-z/2} = \frac{-1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n}.$$

Hence, in this region we have

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{z}\left(\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n}\right) = \frac{1}{z}\sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right)z^n.$$

Multiplying the $1/z$ through the sum and reindexing we have

$$\frac{1}{z(z-1)(z-2)} = \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+2}}\right)z^n + \frac{1}{2z}$$

for $0 < |z| < 1$. Uniqueness of Laurent series guarantees this is the desired expansion.

(b) When $1 < |z| < 2$ we have

$$\frac{-1}{z-1} = \frac{-1}{z}\frac{1}{1-1/z} = \frac{-1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = -\sum_{n=1}^{\infty}\frac{1}{z^n}$$

and, as above,

$$\frac{1}{z-2} = \frac{-1}{2}\frac{1}{1-z/2} = \frac{-1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n}.$$

Thus

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{z}\left(-\sum_{n=1}^{\infty}\frac{1}{z^n} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n}\right) = -\sum_{n=0}^{\infty}\frac{z^{n-1}}{2^{n+1}} - \sum_{n=1}^{\infty}\frac{1}{z^{n+1}}.$$

Reindexing we find

$$\frac{1}{z(z-1)(z-2)} = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+2}} - \frac{1}{2z} - \sum_{n=2}^{\infty}\frac{1}{z^n}$$

and once again the uniqueness of Laurent series tells us that this is the expression we sought.

3.3.8 If f and g are both analytic with zeros of order k at z_0 then we can write $f(z) = (z-z_0)^k\phi(z)$ and $g(z) = (z-z_0)^k\psi(z)$ where ϕ and ψ are analytic wherever f and g are (in particular, in some neighborhood of z_0) and $\phi(z_0) \neq 0$, $\psi(z_0) \neq 0$. It follows that for $z \neq z_0$ we have

$$\frac{f(z)}{g(z)} = \frac{\phi(z)}{\psi(z)}.$$

Since $\psi(z_0) \neq 0$ and both ψ and ϕ are continuous at z_0 we have

$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \lim_{z\to z_0}\frac{\phi(z)}{\psi(z)} = \frac{\phi(z_0)}{\psi(z_0)}.$$

This proves that f/g has a removable singularity at z_0. The problem is then finished by appealing to the following result.

Proposition. Let f be analytic at z_0 with a zero of order k there. Write $f(z) = (z-z_0)^k\phi(z)$. Then $\phi(z)$ is analytic at z_0 and $\phi(z_0) = f^{(k)}(z_0)/k!$.

Proof. We already know that $\phi(z)$ is analytic at z_0. We can therefore write

$$\phi(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

in some neighborhood of z_0. Then

$$f(z) = (z-z_0)^k\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=k}^{\infty} a_{n-k}(z-z_0)^n$$

in some neighborhood of z_0. Applying the uniqueness of Taylor series to this expression we find that

$$a_0 = \frac{f^{(k)}(z_0)}{k!}.$$

On the other hand, from the original expression defining the a_n we know that $a_0 = \phi(z_0)$. The result follows. □

3.3.18 The function $e^{1/z}$ is analytic on $\mathbb{C} \setminus \{0\}$ and therefore has an isolated singularity at $z_0 = 0$. Appealing to the Taylor series for $e^{1/z}$ we find that for $z \neq 0$ we have

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = 1 + \sum_{n=1}^{\infty} \frac{1/n!}{z^n}.$$

By uniqueness, this must be the Laurent series expansion of $e^{1/z}$ on $\mathbb{C} \setminus \{0\}$. However, we know that the Laurent series coefficients are given by

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{e^{1/z}}{z^{n+1}} dz$$

and

$$b_n = \frac{1}{2\pi i} \int_\gamma e^{1/z} z^{n-1} dz.$$

Comparing to the series expression above we find that we must have

$$\frac{1}{2\pi i} \int_\gamma \frac{e^{1/z}}{z^{n+1}} dz = 0$$

for $n \geq 1$,

$$\frac{1}{2\pi i} \int_\gamma \frac{e^{1/z}}{z} dz = 1$$

and

$$\frac{1}{2\pi i} \int_\gamma e^{1/z} z^{n-1} dz = \frac{1}{n!}$$

for $n \geq 1$. Hence

$$\int_\gamma z^n e^{1/z} dz = \begin{cases} 0 & \text{, if } n \leq -2 \\ \frac{2\pi i}{(n+1)!} & \text{, if } n \geq -1. \end{cases}$$

3.R.4 Since e^z is entire, for any $z \in \mathbb{C}$ we have

$$f(z) = e^{z^2} = \sum_{n=0}^{\infty} \frac{(z^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{2^n}}{n!}.$$

Uniqueness of Taylor series guarantees that the expression on the right is the Taylor series for e^{z^2} at the origin. In particular, this means that the coefficient of z^k appearing on the right hand side must be given by $f^{(k)}(0)/k!$. Hence

$$\frac{f^{(68)}(0)}{68!} = \frac{1}{34!}$$

or $f^{(68)}(0) = 68!/34!$.

3.R.12 Since $f(z)$ is analytic for $|z| < 1$, we know that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

for all $|z| < 1$. Let $z \in \mathbb{C}$. Then, according to our hypothesis we have

$$\left| \frac{f^{(n)}(0)}{n!} z^n \right| < \frac{M^n}{n!} |z|^n = \frac{(M|z|)^n}{n!}$$

for every $n \geq 0$. The series

$$\sum_{n=0}^{\infty} \frac{(M|z|)^n}{n!}$$

converges to $e^{M|z|}$. It follows that the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

is absolutely convergent. Since $z \in \mathbb{C}$ was arbitrary, this means that the radius of convergence of the latter series must be infinite and hence that series represents an entire function. Since f agrees with this series for $|z| < 1$, we find that the series provides and extension of f to an entire function.