3.3.16 If f has a zero of multiplicity k at z_0 then we can write $f(z) = (z - z_0)^k \phi(z)$, where ϕ is analytic and $\phi(z_0) \neq 0$. Differentiating this expression yields $f'(z) = k(z-z_0)^{k-1}\phi(z) + (z-z_0)^k\phi'(z)$. Therefore

$$\frac{f'(z)}{f(z)} = \frac{k(z-z_0)^{k-1}\phi(z) + (z-z_0)^k\phi'(z)}{(z-z_0)^k\phi(z)} = \frac{k}{z-z_0} + \frac{\phi'(z)}{\phi(z)}.$$

Since $\phi(z_0) \neq 0$, the function $\phi'(z)/\phi(z)$ is analytic at z_0 and therefore has a convergent Taylor series in a neighborhood of z_0. Thus

$$\frac{f'(z)}{f(z)} = \frac{k}{z-z_0} + \frac{\phi'(z)}{\phi(z)} = \frac{k}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

for all z in a deleted neighborhood of z_0. Uniqueness of such expressions implies that this is the Laurent series for f'/f in a deleted neighborhood of z_0 and therefore f'/f has a simple pole at z_0 with residue k.

3.3.20(a) The closure of a set A is the intersection of all the closed sets that contain A. It is not hard to show that an element belongs to the closure of A if and only if every neighborhood of that point intersects A. Therefore, what we need to prove the following: given any $w \in \mathbb{C}$ and any $\epsilon > 0$, there exists $z \in U$ so that $f(z) \in D(w;\epsilon)$.

So, let $w \in \mathbb{C}$ and $\epsilon > 0$. According to the Casorati-Weierstrass Theorem, there is a sequence $z_1, z_2, z_3, \cdots \in \mathbb{C}$ so that $z_n \to z_0$ and $f(z_n) \to w$. Since $z_0 \in U$ and U is open, there is a $\delta > 0$ so that $D(z_0;\delta) \subset U$. Choose $N_1 \in \mathbb{Z}^+$ so that $|z_n - z_0| < \delta$ for $n \geq N_1$ and choose $N_2 \in \mathbb{Z}^+$ so that $|f(z_n) - w| < \epsilon$. Let $N = \max\{N_1, N_2\}$. Then $z_N \in D(z_0;\delta) \subset U$ and $f(z_N) \in D(w;\epsilon)$. That is, z_N satisfies the required conditions. Since $w \in \mathbb{C}$ and $\epsilon > 0$ were arbitrary, we conclude that the necessary condition holds for all $w \in \mathbb{C}$ and $\epsilon > 0$. Thus, the closure of $f(U)$ is $\mathbb{C}$.

3.R.2 The function $1/\cos z$ fails to be analytic precisely where $\cos z = 0$. The latter occurs if and only if $z = n\pi + \pi/2$ for some $n \in \mathbb{Z}$. Since the derivative of $\cos z$ is $-\sin z$ and $\sin(n\pi + \pi/2) = (-1)^n \neq 0$, we see that $\cos z$ has simple zeros at the points $z = n\pi + \pi/2$, $n \in \mathbb{Z}$. Consequently, $1/\cos z$ has simple poles at these points.

3.R.6(a) The singularities of $f(z)$ occur precisely when $\sin(\pi z) = 0$. The function $\sin(\pi z)$ is zero if and only if $z \in \mathbb{Z}$ and so the singularities of $f(z)$ consist precisely of the integers.

Since the derivative of $\sin(\pi z)$ is $\pi\cos(\pi z)$ and $\pi\cos(\pi n) = (-1)^n\pi \neq 0$ we conclude that the zeros of $\sin(\pi z)$ are all simple. The function $\pi z(1-z^2) = \pi z(1-z)(1+z)$ has simple zeros at $z = 0, \pm 1$, as is evidenced by the given factorization. It follows that

$$f(z) = \frac{\pi z(1-z^2)}{\sin(\pi z)}$$

has simple poles at $z \in \mathbb{Z}$, $z \neq 0, \pm 1$ and that the singularities at $z = 0, \pm 1$ are removable.

3.R.18

(a) If $|z| < 1$ then

$$\frac{1}{1+z^2} + \frac{1}{3-z} = \frac{1}{1+z^2} + \frac{1}{3}\frac{1}{1-z/3} = \sum_{n=0}^{\infty}(-1)^n z^{2n} + \frac{1}{3}\sum_{n=0}^{\infty}\frac{z^n}{3^n}.$$

Since

$$\sum_{n=0}^{\infty}(-1)^n z^{2n} = \sum_{n=0}^{\infty}\left(\frac{(-1)^n+1}{2}\right)(-1)^{n/2}z^n$$

we can combine the series above to get

$$\frac{1}{1+z^2} + \frac{1}{3-z} = \sum_{n=0}^{\infty}\left(\left(\frac{(-1)^n+1}{2}\right)(-1)^{n/2} + \frac{1}{3^{n+1}}\right)z^n,$$

which is valid for $|z| < 1$.

(b) If $1 < |z| < 3$ then

$$\frac{1}{1+z^2} = \frac{1}{z^2}\frac{1}{1+1/z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n}} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n}}$$

since $|1/z^2| < 1$. As above, we still have

$$\frac{1}{3-z} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{z^n}{3^n}$$

and so

$$\frac{1}{1+z^2} + \frac{1}{3-z} = \sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n}}.$$

(c) If $|z| > 3$ then as above we have

$$\frac{1}{1+z^2} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n}}$$

and also

$$\frac{1}{3-z} = \frac{-1}{z}\frac{1}{1-3/z} = \frac{-1}{z}\sum_{n=0}^{\infty}\frac{3^n}{z^n} = -\sum_{n=1}^{\infty}\frac{3^{n-1}}{z^n}.$$

Hence

$$\frac{1}{1+z^2} + \frac{1}{3-z} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n}} - \sum_{n=1}^{\infty}\frac{3^{n-1}}{z^n}$$

and these series may be combined as in part (a) to yield single series.

3.R.20 Let $g(z) = f(1/z)$. Since $1/z$ is analytic for $z \neq 0$ and f is entire, $g(z)$ is analytic for $z \neq 0$. Consequently g has an isolated singularity at $z_0 = 0$. Notice that $\lim z \to 0 g(z) = \lim z \to 0 f(1/z) = \lim w \to \infty f(w) = \infty$. From this it follows that g cannot have a removable singularity at 0 (otherwise the limit would be finite) and that g cannot have an essential singularity at 0 (otherwise the limit could not exist). The only remaining option is that g has a pole at 0. The Laurent expansion of g at 0 then takes the form

$$g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{k}\frac{b_n}{z^n}$$

for some $n \in \mathbb{Z}^+$. Since g is analytic on $\mathbb{C} \setminus \{0\}$, the equality above is valid for all $z \neq 0$. In particular, if $z \neq 0$ then $1/z \neq 0$ and

$$f(z) = g(1/z) = \sum_{n=0}^{\infty}\frac{a_n}{z^n} + \sum_{n=1}^{k} b_n z^n$$

which gives the Laurent expansion for f at 0. Since f is analytic at zero is must be the case that $a_n = 0$ for all $n \geq 1$ (otherwise f would have a pole or essential singularity at 0). But then we have

$$f(z) = a_0 + \sum_{n=1}^{k} b_n z^n.$$

That is, f is a polynomial.

4.1.1

(a) The function $\sin z$ has a simple zero at $z_0 = 0$ since $\sin 0 = 0$ but $\cos 0 = 1 \neq 0$. However, $e^z - 1$ also has a zero at $z_0 = 0$, which means that $(e^z - 1)/\sin z$ has a removable singularity at $z_0 = 0$. Consequently, the residue there is 0.

(b) The function $e^z - 1$ has a simple zero at $z_0 = 0$ since $e^z - 1$ vanishes at that point but its derivative, e^z, does not. Therefore $1/(e^z - 1)$ has a simple pole at $z_0 = 0$ with residue

$$\left.\frac{1}{e^z}\right|_{z=0} = 1$$

by Proposition 4.1.2.

(c) Since $z^2 - 2z = z(z-2)$, we see that $z^2 - 2z$ has a simple zero at $z_0 = 0$. Since $z + 2$ is nonzero at this point, we conclude that $(z+2)/(z^2 - 2z)$ has a simple pole at $z_0 = 0$ with residue

$$\left.\frac{z+2}{2z-2}\right|_{z=0} = -1,$$

again by Proposition 4.1.2.

(d) z^4 has a zero of order 4 at 0 and $e^z + 1$ is nonzero at 0 so that $(e^z+1)/z^4$ must have a pole of order 4 at 0. The residue can be computed using Proposition 4.1.6:

$$\operatorname{Res}\left(\frac{e^z+1}{z^4}; 0\right) = \frac{1}{3!}\left.\frac{d^3}{dz^3}(1+e^z)\right|_{z=0} = \frac{1}{6}.$$

(e) Since $(z^2-1)^2 = (z-1)^2(z+1)^2$, $(z^2-1)^2$ has a double zero at $z_0 = 1$. Since e^z is nonzero at 1, we conclude that

$$\frac{e^z}{(z^2-1)^2}$$

has a double pole at $z_0 = 1$ with residue

$$\left.\frac{d}{dz}(z-1)^2\frac{e^z}{(z^2-1)^2}\right|_{z=1} = \left.\frac{d}{dz}\frac{e^z}{(z+1)^2}\right|_{z=1} = \left.\frac{(z-1)e^z}{(z+1)^3}\right|_{z=1} = 0.$$

4.1.2

(a) Since e^{z^2} is nonzero everywhere, the pole is simple and the residue is

$$\lim_{z\to 1}(z-1)\frac{e^{z^2}}{z-1} = e.$$

(b) The function in question is analytic at 0 and so the residue there is 0.

(c) Let $f(z) = \cos z - 1$. Since $f(0) = 0$, $f'(0) = 0$ and $f''(0) = 1$, we find that f has a double zero at 0. Since $g(z) = z$ has a simple zero at 0, it follows that $f(z)/g(z) = (\cos z - 1)/z$ has a removable singularity at 0 (and that, in fact, when we remove the singularity the function has a simple zero at 0). Therefore $(\cos z - 1)^2/z^2$ also has a removable singularity at 0 and so the residue at 0 is 0.

(d) $i = e^{i\pi/2}$ is a simple zero of $z^4 - 1$ (since it is not a zero of the derivative) and is not a zero of z^2, so it follows that $z^2/(z^4-1)$ has simple zero at i with residue

$$\left.\frac{z^2}{4z^3}\right|_{z=i} = \frac{1}{4i} = -\frac{i}{4}.$$

4.1.8

(a) The zeros of $e^z - 1$ occur at $z = 2n\pi i$, $n \in \mathbb{Z}$, and are all simple since the derivative, e^z, of $e^z - 1$ does not vanish at these points. Therefore we can compute the residue of $1/(e^z - 1)$ at $z = 2n\pi i$ using Proposition 4.1.2:

$$\operatorname{Res}\left(\frac{1}{e^z - 1}; 2n\pi i\right) = \left.\frac{1}{e^z}\right|_{z=2n\pi i} = 1.$$

(b) The chain rule tells us that the only point at which $\sin(1/z)$ fails to be analytic is $z = 0$. Since $\sin z$ is entire it equals its Taylor series centered at the origin at every point. In particular, this means that for $z \neq 0$ we have

$$\sin\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z}\right)^{2n+1}$$

This expression gives the Laurent expansion of $\sin(1/z)$ on $\mathbb{C}\setminus\{0\}$ and shows both that the singularity at $z_0 = 0$ is essential and that the residue at that point is 1.

4.2.2 Let $f(z)$ be analytic on a connected open set A and let γ be any closed curve in A homotopic to a point in A. Let $z_0 \in A$ with $z_0 \notin \gamma$. The function $f(z)/(z - z_0)$ is analytic on $A \setminus \{z_0\}$ with residue $f(z_0)$ at z_0. The Residue Theorem then immediately gives

$$\int_\gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i I(\gamma; z_0) \operatorname{Res}\left(\frac{f(z)}{(z - z_0)}; z_0\right) = 2\pi i I(\gamma; z_0) f(z_0)$$

which is precisely the statement of Cauchy's Integral Formula.

4.2.3 The function $z/(z^2+2z+5)$ has singularities where $z^2+2z+5 = 0$, i.e. at $z = -1 \pm 2i$. Since neither of these points lie inside the unit circle, the function $z/(z^2+2z+5)$ is analytic inside and on the unit circle and so by Cauchy's Theorem we have

$$\int_{|z|=1} \frac{z}{z^2 + 2z + 5}\, dz = 0.$$

4.2.4 According to problem 4.1.8(a), the function $1/(e^z - 1)$ has simple poles of residue 1 at the points $z = 2n\pi i$ for $n \in \mathbb{Z}$. If γ is the circle of radius 9 centered at 0 then γ contains only the poles $0, \pm 2\pi i$. Hence

$$\int_\gamma \frac{dz}{e^z - 1} = 2\pi i(1 + 1 + 1) = 6\pi i.$$

4.2.5 The function $\tan z = \sin z / \cos z$ has singularities where $\cos z = 0$, i.e. at the points $z = \pi/2 + n\pi$ for $n \in \mathbb{Z}$. Since $\sin z$ is nonzero at these points we conclude that these are all simple zeros of $\cos z$ and simple poles of $\sin z / \cos z$. For $n \in \mathbb{Z}$ we have

$$\operatorname{Res}\left(\frac{\sin z}{\cos z}; \frac{\pi}{2} + n\pi\right) = \frac{\sin(\pi/2 + n\pi)}{-\sin(\pi/2 + n\pi)} = -1.$$

If γ is the circle of radius 8 centered at 0 then γ contains only the singularities $\pi/2 + n\pi$ for $n = -3, -2, -1, 0, 1, 2$ and therefore

$$\int_\gamma \tan z \, dz = 2\pi i(-1 + -1 + -1 + -1 + -1 + -1) = -12\pi i.$$

4.2.6 The singularities of $(5z - 2)/z(z - 1)$ occur at $z = 0$ and $z = 1$ and since these are both simple zeros of the denominator (but not zeros of the numerator) these are both simple poles. The residues are

$$\operatorname{Res}\left(\frac{5z - 2}{z(z - 1)}; 0\right) = \frac{-2}{-1} = 2$$

and

$$\operatorname{Res}\left(\frac{5z - 2}{z(z - 1)}; 1\right) = \frac{3}{1} = 3.$$

Therefore, if γ is any circle which contains both $z = 0$ and $z = 1$ then

$$\int_\gamma \frac{5z - 2}{z(z - 1)} \, dz = 2\pi i(2 + 3) = 10\pi i.$$