Complex Analysis
Fall 2007 Homework 1: Solutions

1.1.2.

(a) $(2+3i)(4+i) = (8-3)+(12+2)i = 5+14i$

(b) $(8+6i)^2 = (64-36)+(48+48)i = 28+96i$

(c)

$$\begin{aligned}
\left(1+\frac{3}{1+i}\right)^2 &= \left(1+\frac{3(1-i)}{(1+i)(1-i)}\right)^2 \\
&= \left(1+\frac{3-3i}{2}\right)^2 \\
&= \left(\frac{5}{2}-\frac{3}{2}i\right)^2 \\
&= \left(\frac{25}{4}-\frac{9}{4}\right)+\left(-\frac{15}{4}-\frac{15}{4}\right)i \\
&= 4-\frac{15}{2}i
\end{aligned}$$

1.1.6.

(a) If $z = x+iy$ we have

$$\begin{aligned}
\frac{z+1}{2z-5} &= \frac{(z+1)(2\bar{z}-5)}{(2z-5)(2\bar{z}-5)} \\
&= \frac{2z\bar{z}-5z+2\bar{z}-5}{4z\bar{z}-10(z+\bar{z})+25} \\
&= \frac{2|z|^2-5z+2\bar{z}-5}{4|z|^2-10(z+\bar{z})+25} \\
&= \frac{2(x^2+y^2)-5x+2x-5+-5yi-2yi}{4(x^2+y^2)-20x+25} \\
&= \frac{2(x^2+y^2)-3x-5}{4(x^2+y^2)-20x+25}+\frac{-7y}{4(x^2+y^2)-20x+25}i
\end{aligned}$$

so that

$$\mathrm{Re}\left(\frac{z+1}{2z-5}\right) = \frac{2(x^2+y^2)-3x-5}{4(x^2+y^2)-20x+25} \quad , \quad \mathrm{Im}\left(\frac{z+1}{2z-5}\right) = \frac{-7y}{4(x^2+y^2)-20x+25}.$$

(b) If $z = x + iy$ then

$$z^3 = (x+iy)^3 = x^3 + 3x^2yi + 3xy^2i^2 + y^3i^3 = (x^3 - 3xy^2) + (3x^2y - y^3)i$$

so that

$$\text{Re}z^3 = x^3 - 3xy^2 \quad , \quad \text{Im}z^3 = 3x^2y - y^3.$$

1.1.18.

(a) $(1-i)^{-1} = \frac{1+i}{(1-i)(1+i)} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i$

(b) $\frac{1+i}{1-i} = (1+i)(1-i)^{-1} = (1+i)\left(\frac{1}{2} + \frac{1}{2}i\right) = \left(\frac{1}{2} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{2}\right)i = i$

1.2.2.

(a) The equation $z^6 + 8 = 0$ is equivalent to $z^6 = -8$. Since $|-8| = 8$ and $\arg(-8) = \pi$, the solutions to the latter equation are

$$z_k = \sqrt[6]{8}\left(\cos\left(\frac{\pi}{6} + \frac{\pi k}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{\pi k}{3}\right)\right)$$

for $k = 0, 1, \ldots, 5$.

(b) The equation $z^3 - 4 = 0$ is equivalent to $z^3 = 4$ which, since $|4| = 4$ and $\arg(4) = 0$, has the solutions

$$z_k = \sqrt[3]{4}\left(\cos\left(\frac{2\pi k}{3}\right) + i\sin\left(\frac{2\pi k}{3}\right)\right)$$

for $k = 0, 1, 2$.

1.2.4. Recalling that conjugation preserves the arithmetic of $\mathbb{C}$, we have

$$\overline{\left(\frac{(8-2i)^{10}}{(4+6i)^5}\right)} = \frac{(8+2i)^{10}}{(4-6i)^5}.$$

1.2.6. DeMoivre's formula and the binomial theorem give

$$\begin{aligned} \cos 6x + i\sin 6x &= (\cos x + i\sin x)^6 \\ &= (\cos^6 x - 15\cos^4 x\sin^2 x + 15\cos^2 x\sin^4 x - \sin^6 x) \\ &\quad + i(6\cos^5 x\sin x - 20\cos^3 x\sin^3 x + 6\cos x\sin^5 x). \end{aligned}$$

Equating real and imaginary parts gives

$$\begin{aligned} \cos 6x &= \cos^6 x - 15\cos^4 x\sin^2 x + 15\cos^2 x\sin^4 x - \sin^6 x \\ \sin 6x &= 6\cos^5 x\sin x - 20\cos^3 x\sin^3 x + 6\cos x\sin^5 x. \end{aligned}$$

1.2.8. Since $|\cdot|$ preserves multiplication and division we have

$$\left|\frac{(2-3i)^2}{(8+6i)^2}\right| = \frac{|2-3i|^2}{|8+6i|^2} = \frac{4+9}{64+36} = \frac{13}{100}$$

1.2.10. Let

$$p(z) = \sum_{j=0}^{n} a_j z^j$$

be a polynomial with $a_j \in \mathbb{R}$ for all j. To say that the roots of $p(z)$ "occur in complex pairs" means that if $z_0 \in \mathbb{C}$ is a root, then so too is $\overline{z_0}$. To see that this is the case, let $z_0 \in \mathbb{C}$ with $p(z_0) = 0$. Then, since conjugation preserves arithmetic and $a_j = \overline{a_j}$ for all j,

$$0 = \overline{0} = \overline{p(z_0)} = \overline{\sum_{j=0}^{n} a_j z_0^j} = \sum_{j=0}^{n} \overline{a_j}\, \overline{z_0}^j = \sum_{j=0}^{n} a_j\, \overline{z_0}^j = p(\overline{z_0})$$

1.2.14. Notice first that

$$\left|\frac{z-w}{1-z\bar{w}}\right| = \frac{|z-w|}{|1-z\bar{w}|} = \frac{|w-z|}{|1-\bar{z}w|} = \left|\frac{w-z}{1-w\bar{z}}\right|.$$

That is, the expression in question is symmetric in z and w. We may therefore assume, without loss of generality, that $|w| = 1$. The second, and perhaps more important observation, is that if $|w| = 1$ then $w^{-1} = \bar{w}$, since $1 = |w|^2 = w\bar{w}$. We therefore have

$$\left|\frac{z-w}{1-z\bar{w}}\right| = \left|\frac{z-w}{1-zw^{-1}}\right| = \left|\frac{z-w}{(w-z)w^{-1}}\right| = \left|\frac{w(z-w)}{w-z}\right| = \frac{|w||z-w|}{|w-z|} = 1$$

which is what we wanted to show.

1.2.18.

(a) Since $|z|^2 \in \mathbb{R}$, $|z|^2 = z\bar{z}$ and congruence mod 2π is an equivalence relation, if $z \neq 0$ we have

$$\begin{aligned} 0 &\equiv \arg|z|^2 \pmod{2\pi} \\ &\equiv \arg z\bar{z} \pmod{2\pi} \\ &\equiv \arg z + \arg\bar{z} \pmod{2\pi} \end{aligned}$$

which shows that $\arg\bar{z} \equiv -\arg z \pmod{2\pi}$.

(b) As above, we have

$$\begin{aligned} \arg z &\equiv \arg(zw/w) \pmod{2\pi} \\ &\equiv \arg(z/w) + \arg w \pmod{2\pi} \end{aligned}$$

which shows that $\arg(z/w) \equiv \arg z - \arg w \pmod{2\pi}$.

(c) Let $z = x + iy \in \mathbb{C}$. Then $|z|^2 = x^2 + y^2$ so that $|z| = 0$ iff $|z|^2 = 0$ iff $x^2 + y^2 = 0$. Since $x, y \in \mathbb{R}$, this can happen iff $x = y = 0$ iff $z = 0$.

1.2.24. Since cubing a complex number triples its argument and cubes its modulus, it is clear that z^3 maps the disk $\{z \in \mathbb{C} \mid |z| < 2\}$ onto the disk $\{w \in \mathbb{C} \mid |w| < 8\}$. Multiplication by i rotates this disk by $\pi/4$ radians back onto itself and adding 1 shifts its radius to the point 1 on the real axis. Since the right-most edge of the shifted disk passes through the point 9, we find that $\sup_{|z|<2} \mathrm{Re}(iz^3 + 1) = 9$.

1.3.2.

(a) $e^{3-i} = e^3(\cos(-1) + i\sin(-1)) = e^3\cos 1 - ie^3 \sin 1$

(b)

$$\begin{aligned}
\cos(2+3i) &= \frac{e^{i(2+3i)} + e^{-i(2+3i)}}{2} \\
&= \frac{e^{-3+2i} + e^{3-2i}}{2} \\
&= \frac{e^{-3}(\cos 2 + i\sin 2) + e^3(\cos(-2) + i\sin(-2))}{2} \\
&= \cos 2 \frac{(e^3 + e^{-3})}{2} - i \sin 2 \frac{e^3 - e^{-3}}{2} \\
&= \cos 2 \cosh 3 - i \sin 2 \sinh 3
\end{aligned}$$

1.3.4.

(a) If $\sin z = (3+i)/4$ then $4\sin z = 3 + i$. The definition of $\sin z$ then gives

$$-2i(e^{iz} - e^{-iz}) = 3 + i$$

which, upon multiplication by e^{iz} (which is nonzero), is equivalent to

$$-2i(e^{iz})^2 + 2i = (3+i)e^{iz}$$

or

$$-2i(e^{iz})^2 - (3+i)e^{iz} + 2i = 0.$$

The quadratic formula yields

$$e^{iz} = \frac{(3 + i \pm \sqrt{(3+i)^2 - 16})}{-4i} = \frac{3 + i \pm \sqrt{-8+6i}}{-4i}.$$

At this point we need to compute the square root. Writing $-8 + 6i = r(\cos\theta + i\sin\theta)$ $(r > 0)$ we find that $r = 10$ and $\cos\theta = -8/10$. Using the standard half-angle formulas, we get $\cos(\theta/2) = 1/\sqrt{10}$ and $\sin(\theta/2) = 3/\sqrt{10}$, so that one of the square roots of $-8 + 6i$ is

$$\sqrt{r}(\cos(\theta/2) + i\sin(\theta/2)) = \sqrt{10}(1/\sqrt{10} + i3/\sqrt{10}) = 1 + 3i.$$

Therefore, our solutions satisfy

$$e^{iz} = \frac{3+i \pm \sqrt{-8+6i}}{-4i} = \frac{4+4i}{-4i}, \frac{2-2i}{-4i} = -1+i, \frac{1}{2}+\frac{1}{2}i.$$

Let $\log z$ denote the branch of the logarithm with imaginary part in the interval $[-\pi, pi)$. Applying this branch to the equation above we find that all of the solutions are given by

$$\begin{aligned} iz &= \log(-1+i) + 2n\pi i \\ iz &= \log\left(\frac{1}{2}+\frac{1}{2}i\right) + 2m\pi i \end{aligned}$$

for $m, n \in \mathbb{Z}$. Since $\log(-1+i) = \log\sqrt{2} + 3\pi i/4$ and $\log((1+i)/2) = \log(1/\sqrt{2}) + \pi i/4 = -\log\sqrt{2} + \pi i/4$ the two equations above can be rewritten as

$$\begin{aligned} z &= 2n\pi + \frac{3\pi}{4} - i\log\sqrt{2} \\ z &= 2m\pi + \frac{\pi}{4} + i\log\sqrt{2}. \end{aligned}$$

Letting m, n vary through all possible integer values yields the complete set of solutions to the original equation.

(b) The procedure is exactly the same as that above. Plugging in the definition of $\sin z$, the equation $\sin z = 4$ becomes the equation $(e^{iz})^2 - 8ie^{iz} - 1 = 0$, which the quadratic formula transforms to

$$e^{iz} = (4 \pm \sqrt{15})i.$$

Using the same branch of $\log z$ as above we get

$$iz = \log((4 \pm \sqrt{15})i) + 2n\pi i = \log(4 \pm \sqrt{15}) + i\left(\frac{\pi}{2} + 2n\pi\right)$$

where $n \in \mathbb{Z}$ is arbitrary. Dividing by i gives the final answer:

$$z = \frac{\pi}{2} + 2n\pi - i\log\sqrt{(4 \pm \sqrt{15})}$$

for any $n \in \mathbb{Z}$.

1.3.14. If $w = e^{x+iy} = e^x e^{iy}$ then $|w| = e^x$ and $\arg w \equiv y \pmod{2\pi}$. Hence, if y is fixed, e^z lies on a ray emanating from the origin, making an angle of y with the real axis. As $x \to \infty$, $|w| = e^x \to \infty$ indicating that e^z moves out along this ray away from the origin indefinitely. On the other hand, as $x \to -\infty$, $|w| = e^x \to 0$ so that e^z moves along the ray closer and closer to the origin.

If x is fixed, then $|w|$ is a constant, indicating that e^z remains on the circle of radius e^x centered at 0. Since $\arg w \equiv y \pmod{2\pi}$, as $y \to \infty$ the point $w = e^z$ simply moves around this circle repeatedly counterclockwise and as $y \to -\infty$ the circle is traced out infinitely often in the clockwise direction.

1.3.24. We start by noting that if $z = x + iy$ then

$$\begin{aligned}\cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ &= \frac{e^{-y}(\cos x + i \sin x) + e^{y}(\cos(-x) + i \sin(-x))}{2} \\ &= \cos x \cosh y - i \sin x \sinh y.\end{aligned}$$

A horizontal line has the form $z = x + iy_0$ where $y_0 \in \mathbb{R}$ is fixed and $x \in \mathbb{R}$ varies. Writing $\cos z = u + iv$ we find that the real and imaginary parts of the image of this horizontal line satisfy

$$\begin{aligned}u &= \cos x \cosh y_0 \\ v &= -\sin x \sinh y_0.\end{aligned}$$

For variable x, this is the standard clockwise parametrization of the ellipse

$$\frac{u^2}{\cosh^2 y_0} + \frac{v^2}{\sinh^2 y_0} = 1.$$ [1]

That is, horizontal lines are mapped onto ellipses.

A vertical line is given by $z = x_0 + iy$ where $x_0 \in \mathbb{R}$ is fixed and $y \in \mathbb{R}$ is free to vary. In this case we have, again writing $\cos z = u + iv$,

$$\begin{aligned}u &= \cos x_0 \cosh y \\ v &= -\sin x_0 \sinh y\end{aligned}$$

which as $y \in \mathbb{R}$ varies gives the standard parametrization of the hyperbola

$$\frac{u^2}{\cos^2 x_0} - \frac{v^2}{\sin^2 x_0} = 1.$$ [2]

That is, vertical lines are mapped onto hyperbolas.

[1] Provided $\sinh y_0 \neq 0$. It is left to the reader to describe the image in the case $\sinh y_0 = 0$.

[2] Provided that $\cos x_0$ and $\sin x_0$ are both nonzero. It is left to the reader to determine the image when $\cos x_0 = 0$ or $\sin x_0 = 0$.