1.3.6.

(a) We have $\log|-i| = \log 1 = 0$ and $\arg(-i) \in \{-\pi/2 + 2n\pi \,|\, n \in \mathbb{Z}\}$. Hence, the values of $\log(-i)$ are

$$i\left(\frac{-\pi}{2} + 2n\pi\right)$$

for $n \in \mathbb{Z}$.

(b) We have $\log|1+i| = \log\sqrt{2} = (1/2)\log 2$ and $\arg(1+i) \in \{\pi/4 + 2n\pi \,|\, n \in \mathbb{Z}\}$. Hence, the values of $\log(1+i)$ are

$$\frac{1}{2}\log 2 + i\left(\frac{\pi}{4} + 2n\pi\right).$$

1.3.8.

(a) Since i is not a rational number, we know that $(-1)^i$ has infinitely many values. To compute them we first compute the values of $\log(-1)$. Since $\log|-1| = \log 1 = 0$ and $\arg(-1) \in \{\pi + 2n\pi \,|\, n \in \mathbb{Z}\}$ the values of $\log(-1)$ are $i(\pi + 2n\pi)$ for $n \in \mathbb{Z}$. Hence, the values of $(-1)^i$ are

$$e^{i\log(-1)} = e^{i^2(\pi+2n\pi)} = e^{-(\pi+2n\pi)}$$

for $n \in \mathbb{Z}$.

(b) Again, i is not rational so we expect infinitely many values for 2^i. As above, we have $\log|2| = \log 2$ and $\arg 2 \in \{2n\pi \,|\, n \in \mathbb{Z}\}$ so that $\log 2 = \log 2 + 2n\pi i$ and

$$2^i = e^{i\log 2} = e^{-2n\pi + i\log 2}$$

for $n \in \mathbb{Z}$.

1.3.12. We begin by observing that for any $\alpha \in \mathbb{C}$ we have (by definition)

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = -i\frac{e^{i\alpha} - e^{-i\alpha}}{e^{i\alpha} + e^{-i\alpha}}.$$

If we multiply the numerator and denominator by $e^{i\alpha}$ this becomes

$$\tan\alpha = -i\frac{(e^{i\alpha})^2 - 1}{(e^{i\alpha})^2 + 1}.$$

Given $z \in \mathbb{C} \setminus \{i, -i\}$[1], let

$$\left(\frac{1+iz}{1-iz}\right)^{1/2}$$

[1]While it is only implicit in this problem, one can easily show that the equation $\tan w = i$ has no solution for $w \in \mathbb{C}$. This exercise, therefore, proves that the function $f(z) = \tan z$ maps $\mathbb{C}$ onto the set $\mathbb{C} \setminus \{i, -i\}$

denote either square root of $(1+iz)/(1-iz)$ and let

$$\log\left(\frac{1+iz}{1-iz}\right)^{1/2}$$

denote any fixed value of the logarithm of $((1+iz)/(1-iz))^{1/2}$. Set

$$\alpha = \frac{1}{i}\log\left(\frac{1+iz}{1-iz}\right)^{1/2}.$$

Then, since $e^{\log w} = w$ for any nonzero w and any branch of the logarithm we have

$$e^{i\alpha} = \exp\left(\log\left(\frac{1+iz}{1-iz}\right)^{1/2}\right) = \left(\frac{1+iz}{1-iz}\right)^{1/2}.$$

But $(w^{1/2})^2 = w$ for any w and any choice of the square root, so the above implies that

$$(e^{i\alpha})^2 = \left(\left(\frac{1+iz}{1-iz}\right)^{1/2}\right)^2 = \frac{1+iz}{1-iz}.$$

Plugging this into our expression for $\tan\alpha$ above yields

$$\tan\alpha = -i\frac{\left(\frac{1+iz}{1-iz}\right)-1}{\left(\frac{1+iz}{1-iz}\right)+1} = -i\frac{(1+iz)-(1-iz)}{(1+iz)+(1-iz)} = -i\frac{2iz}{2} = z$$

which is exactly what we needed to show. Since our choice of square root and logarithm were arbitrary, the identity holds for all branches.

1.3.18. Let $b \in \mathbb{R}$. For $a \in \mathbb{C}\setminus\{0\}$ let $\log a = \log|a| + i\arg a$ for some branch of $\arg a$. Then

$$|a^b| = |e^{b\log a}| = |e^{b\log|a|}e^{ib\arg a}| = |e^{b\log|a|}|\cdot|e^{ib\arg a}|.$$

Since $b \in \mathbb{R}$, $b\log|a| \in \mathbb{R}$ and $b\arg a \in \mathbb{R}$. Therefore,

$$e^{b\log|a|} = (e^{\log|a|})b = |a|^b$$

and

$$|e^{ib\arg a}| = 1.$$

Combining this with what we had above we get

$$|a^b| = |a|^b$$

which is what we sought to show. Note that since our choice of branch of the logarithm was arbitrary, this identity holds for any branch of a^b.

1.3.26a. Any line parallel to the real axis can be described parametrically by $z = t + iy_0$, where $y_0 \in \mathbb{R}$ is fixed and $t \in \mathbb{R}$ is arbitrary. Writing $z^2 = u + iv$ we find that $u = t^2 - y_0^2$ and $v = 2y_0t$. Solving the latter for t and substituting this into the former yields

$$u = \left(\frac{v}{2y_0}\right)^2 - y_0^2 = \left(\frac{v}{2y_0} - y_0\right)\left(\frac{v}{2y_0} + y_0\right)$$

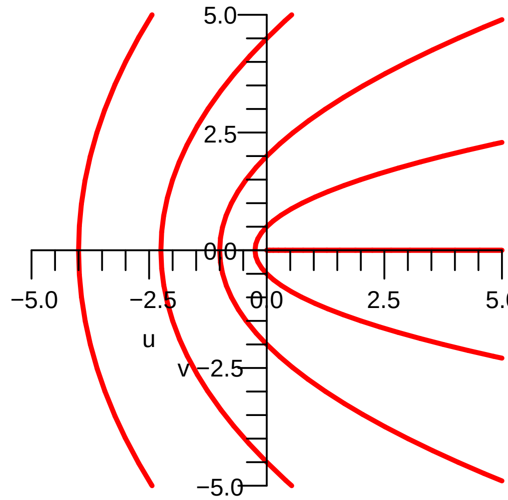

Figure 1: The images of the horizontal lines $y = 0, 1/2, 1, 3/2$ and 2 (from right to left) under the mapping $z \mapsto z^2$.

which shows that the points (u, v) lie on a right-ward opening parabola (unless $y_0 = 0$; in that case the points lie on the non-negative real axis). The fact that we get every point on this parabola as z moves along the original line follows from the fact that as t varies through all real numbers, so too does $v = 2y_0 t$.

Additional problem.

(a) The first quadrant is the same as $W_{\pi/2}$. The upper half plane is W_π. The entire plane is $W_{2\pi}$.

(b) Let $z \in W_{\theta_0}$. Then we can write $z = re^{i\theta}$ for some $r \geq 0$ and $\theta \in [0, \theta_0]$ and $z^n = r^n e^{in\theta}$. Since $r^n \geq 0$ and $n\theta \in [0, n\theta_0]$, it follows that $z^n \in W_{n\theta_0}$. That is, if $f(z) = z^n$ then $f(W_{\theta_0}) \subset W_{n\theta_0}$.

To prove that f maps W_{θ_0} onto $W_{n\theta_0}$ we must show that, in fact, $f(W_{\theta_0}) = W_{n\theta_0}$. To that end, let $w \in W_{n\theta_0}$. Then we can write $w = re^{i\theta}$ with $r \geq 0$ and $\theta \in [0, n\theta_0]$. Since $r^{1/n} \geq 0$ and $\theta/n \in [0, \theta_0]$, we see that $z = r^{1/n}e^{i\theta/n} \in W_{\theta_0}$. Furthermore, $f(z) = (r^{1/n}e^{i\theta/n})^n = re^{i\theta} = w$. Since $w \in W_{n\theta_0}$ was arbitrary this proves that f maps W_{θ_0} onto $W_{n\theta_0}$.

To prove that f is one-to-one on W_{θ_0} we show that if $z_1, z_2 \in W_{\theta_0}$ and $f(z_1) = f(z_2)$ then $z_1 = z_2$. So, let $z_1, z_2 \in W_{\theta_0}$ and suppose $f(z_1) = f(z_2)$. If $z_1 = 0$ then $0 = f(z_1) = f(z_2) = z_2^n$ which implies that $z_2 = 0 = z_1$. We have the same conclusion if $z_2 = 0$. Therefore we can assume that $z_1, z_2 \neq 0$. In this case, we can write $z_k = r_k e^{i\theta_k}$ with $r_k > 0$ and $\theta_k \in [0, \theta_0]$ for $k = 1, 2$. Then $f(z_1) = f(z_2)$ implies $r_1^n e^{in\theta_1} = r_2^n e^{in\theta_2}$. Because $r_1^n, r_2^n > 0$, uniqueness of polar representations implies that $r_1^n = r_2^n$ and $n\theta_1 \equiv n\theta_2 \pmod{2\pi}$. The first equation immediately gives $r_1 = r_2$. As to the second, it implies that $n\theta_1 - n\theta_2$ is a multiple of 2π. But $n\theta_1, n\theta_n \in [0, n\theta_0] \subset [0, 2\pi)$ so that

$|n\theta_1 - n\theta_2| < 2\pi$. It follows that $n\theta_2 - n\theta_1 = 0$ and hence that $\theta_1 = \theta_2$. Therefore, $z_1 = z_2$. Since $z_1, z_2 \in W_{\theta_0}$ were arbitrary, we have proven that f is one-to-one.

(c) If $\theta_0 = 2\pi/n$ then f is still onto. In fact, the proof of "onto" in part (b) still applies. However, f is no longer one-to-one. For example, if $z_1 = 1$ and $z_2 = e^{2\pi/n}$ then $z_1, z_2 \in W_{2\pi/n}$ and $f(z_1) = f(z_2) = 1$, but $z_1 \neq z_2$.