1.3.6.

(a) We have \( \log |-i| = \log 1 = 0 \) and \( \arg(-i) \in \{-\pi/2 + 2n\pi \mid n \in \mathbb{Z}\} \). Hence, the values of \( \log(-i) \) are
\[
i \left( -\frac{\pi}{2} + 2n\pi \right)
\]
for \( n \in \mathbb{Z} \).

(b) We have \( \log |1+i| = \log \sqrt{2} = (1/2) \log 2 \) and \( \arg(1+i) \in \{\pi/4 + 2n\pi \mid n \in \mathbb{Z}\} \). Hence, the values of \( \log(1+i) \) are
\[
\frac{1}{2} \log 2 + i \left( \frac{\pi}{4} + 2n\pi \right).
\]

1.3.8.

(a) Since \( i \) is not a rational number, we know that \((-1)^i\) has infinitely many values. To compute them we first compute the values of \( \log(-1) \). Since \( \log |-1| = \log 1 = 0 \) and \( \arg(-1) \in \{\pi + 2n\pi \mid n \in \mathbb{Z}\} \) the values of \( \log(-1) \) are \( i(\pi+2n\pi) \) for \( n \in \mathbb{Z} \). Hence, the values of \((-1)^i\) are
\[
e^{i \log(-1)} = e^{i(\pi+2n\pi)} = e^{-(\pi+2n\pi)}
\]
for \( n \in \mathbb{Z} \).

(b) Again, \( i \) is not rational so we expect infinitely many values for \( 2^i \). As above, we have \( \log |2| = \log 2 \) and \( \arg 2 \in \{2n\pi \mid n \in \mathbb{Z}\} \) so that \( \log 2 = \log 2 + 2n\pi i \) and
\[
2^i = e^{i \log 2} = e^{-2n\pi + i \log 2}
\]
for \( n \in \mathbb{Z} \).

1.3.12. We begin by observing that for any \( \alpha \in \mathbb{C} \) we have (by definition)
\[
\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = -i \frac{e^{i\alpha} - e^{-i\alpha}}{e^{i\alpha} + e^{-i\alpha}}.
\]
If we multiply the numerator and denominator by \( e^{i\alpha} \) this becomes
\[
\tan \alpha = -i \frac{(e^{i\alpha})^2 - 1}{(e^{i\alpha})^2 + 1}.
\]
Given \( z \in \mathbb{C} \setminus \{i, -i\} \), let
\[
\left( \frac{1 + iz}{1 - iz} \right)^{1/2}
\]

\(^1\)While it is only implicit in this problem, one can easily show that the equation \( \tan w = i \) has no solution for \( w \in \mathbb{C} \). This exercise, therefore, proves that the function \( f(z) = \tan z \) maps \( \mathbb{C} \) onto the set \( \mathbb{C} \setminus \{i, -i\} \).
denote either square root of $(1 + iz)/(1 - iz)$ and let
\[ \log \left( \frac{1 + iz}{1 - iz} \right)^{1/2} \]
denote any fixed value of the logarithm of $((1 + iz)/(1 - iz))^{1/2}$. Set
\[ \alpha = \frac{1}{i} \log \left( \frac{1 + iz}{1 - iz} \right)^{1/2}. \]

Then, since $e^{\log w} = w$ for any nonzero $w$ and any branch of the logarithm we have
\[ e^{i\alpha} = \exp \left( \log \left( \frac{1 + iz}{1 - iz} \right)^{1/2} \right) = \left( \frac{1 + iz}{1 - iz} \right)^{1/2}. \]

But $(w^{1/2})^2 = w$ for any $w$ and any choice of the square root, so the above implies that
\[ (e^{i\alpha})^2 = \left( \frac{1 + iz}{1 - iz} \right) = \frac{1 + iz}{1 - iz}. \]

Plugging this into our expression for $\tan \alpha$ above yields
\[ \tan \alpha = -\frac{(1 + iz)}{(1 - iz)} - 1 = -\frac{(1 + iz) - (1 - iz)}{(1 + iz) + (1 - iz)} = -\frac{2iz}{2} = z \]
which is exactly what we needed to show. Since our choice of square root and logarithm were arbitrary, the identity holds for all branches.

1.3.18. Let $b \in \mathbb{R}$. For $a \in \mathbb{C} \setminus \{0\}$ let $\log a = \log |a| + i \arg a$ for some branch of $\arg a$. Then
\[ |a^b| = |e^{b \log a}| = |e^{b \log |a|} \cdot e^{ib \arg a}| = |e^{b \log |a|}| \cdot |e^{ib \arg a}|. \]

Since $b \in \mathbb{R}$, $b \log |a| \in \mathbb{R}$ and $b \arg a \in \mathbb{R}$. Therefore,
\[ e^{b \log |a|} = (e^{\log |a|})^b = |a|^b \]
and
\[ |e^{ib \arg a}| = 1. \]

Combining this with what we had above we get
\[ |a^b| = |a|^b \]
which is what we sought to show. Note that since our choice of branch of the logarithm was arbitrary, this identity holds for any branch of $a^b$.

1.3.26a. Any line parallel to the real axis can be described parametrically by $z = t + iy_0$, where $y_0 \in \mathbb{R}$ is fixed and $t \in \mathbb{R}$ is arbitrary. Writing $z^2 = u + iv$ we find that $u = t^2 - y_0^2$ and $v = 2y_0t$. Solving the latter for $t$ and substituting this into the former yields
\[ u = \left( \frac{v}{2y_0} \right)^2 - y_0^2 = \left( \frac{v}{2y_0} - y_0 \right) \left( \frac{v}{2y_0} + y_0 \right) \]
which shows that the points \((u, v)\) lie on a right-ward opening parabola (unless \(y_0 = 0\); in that case the points lie on the non-negative real axis). The fact that we get every point on this parabola as \(z\) moves along the original line follows from the fact that as \(t\) varies through all real numbers, so too does \(v = 2y_0t\).

**Additional problem.**

(a) The first quadrant is the same as \(W_{\pi/2}\). The upper half plane is \(W_\pi\). The entire plane is \(W_{2\pi}\).

(b) Let \(z \in W_{\theta_0}\). Then we can write \(z = re^{i\theta}\) for some \(r \geq 0\) and \(\theta \in [0, \theta_0]\) and \(z^n = r^n e^{i n \theta}\). Since \(r^n \geq 0\) and \(n\theta \in [0, n\theta_0]\), it follows that \(z^n \in W_{n\theta_0}\). That is, if \(f(z) = z^n\) then \(f(W_{\theta_0}) \subset W_{n\theta_0}\).

To prove that \(f\) maps \(W_{\theta_0}\) onto \(W_{n\theta_0}\) we must show that, in fact, \(f(W_{\theta_0}) = W_{n\theta_0}\). To that end, let \(w \in W_{n\theta_0}\). Then we can write \(w = re^{i\theta}\) with \(r \geq 0\) and \(\theta \in [0, n\theta_0]\). Since \(r^{1/n} \geq 0\) and \(\theta/n \in [0, \theta_0]\), we see that \(z = r^{1/n} e^{i \theta/n} \in W_{\theta_0}\). Furthermore, \(f(z) = (r^{1/n} e^{i \theta/n})^n = re^{i\theta} = w\). Since \(w \in W_{n\theta_0}\) was arbitrary this proves that \(f\) maps \(W_{\theta_0}\) onto \(W_{n\theta_0}\).

To prove that \(f\) is one-to-one on \(W_{\theta_0}\) we show that if \(z_1, z_2 \in W_{\theta_0}\) and \(f(z_1) = f(z_2)\) then \(z_1 = z_2\). So, let \(z_1, z_2 \in W_{\theta_0}\) and suppose \(f(z_1) = f(z_2)\). If \(z_1 = 0\) then \(0 = f(z_1) = f(z_2) = z_2^n\) which implies that \(z_2 = 0 = z_1\). We have the same conclusion if \(z_2 = 0\). Therefore we can assume that \(z_1, z_2 \neq 0\). In this case, we can write \(z_k = r_k e^{i \theta_k}\) with \(r_k > 0\) and \(\theta_k \in [0, \theta_0]\) for \(k = 1, 2\). Then \(f(z_1) = f(z_2)\) implies \(r_1^n e^{i n \theta_1} = r_2^n e^{i n \theta_2}\). Because \(r_1^n, r_2^n > 0\), uniqueness of polar representations implies that \(r_1^n = r_2^n\) and \(n\theta_1 \equiv n\theta_2 \pmod{2\pi}\). The first equation immediately gives \(r_1 = r_2\). As to the second, it implies that \(n\theta_1 - n\theta_2\) is a multiple of \(2\pi\). But \(n\theta_1, n\theta_n \in [0, n\theta_0] \subset [0, 2\pi]\) so that...
\[|n\theta_1 - n\theta_2| < 2\pi.\] It follows that \(n\theta_2 - n\theta_1 = 0\) and hence that \(\theta_1 = \theta_2\). Therefore, \(z_1 = z_2\). Since \(z_1, z_2 \in W_{\theta_0}\) were arbitrary, we have proven that \(f\) is one-to-one.

(c) If \(\theta_0 = 2\pi/n\) then \(f\) is still onto. In fact, the proof of “onto” in part (b) still applies. However, \(f\) is no longer one-to-one. For example, if \(z_1 = 1\) and \(z_2 = e^{2\pi/n}\) then \(z_1, z_2 \in W_{2\pi/n}\) and \(f(z_1) = f(z_2) = 1\), but \(z_1 \neq z_2\).