1.3.10. **Claim:** Using the branch of the square root function given in the problem, \( \sqrt{z^2} = z \) iff \( z = 0 \) or \( z = re^{i\theta} \) with \( r > 0 \) and \( 0 \leq \theta < \pi \).

**Proof:** (\( \Leftarrow \)) Suppose \( z = re^{i\theta} \) with \( r \geq 0 \) and \( 0 \leq \theta < \pi \). Then \( z^2 = r^2e^{2i\theta} \) and \( 0 \leq 2\theta < 2\pi \) so that
\[
\sqrt{z^2} = (r^2)^{1/2}e^{i2\theta/2} = re^{i\theta} = z.
\]

(\( \Rightarrow \)) Suppose \( z \) is not of the desired form. We will show that \( \sqrt{z^2} \neq z \). In this case \( z = re^{i\theta} \) with \( r > 0 \) and \( \pi \leq \theta < 2\pi \). Then \( z^2 = r^2e^{i2\theta} = r^2e^{i(2\theta - 2\pi)} \) and since \( 0 \leq 2\theta - 2\pi < 2\pi \) we have
\[
\sqrt{z^2} = (r^2)^{1/2}e^{i(2\theta - 2\pi)/2} = re^{i\theta}e^{i\pi} = -z.
\]
Since \( z \neq 0 \), this is not the same as \( z \).

1.3.34. If \( z = x + iy \), it is shown on page 38 of our text that
\[
\sin z = \sin x \cosh y + i \cos x \sinh y.
\]
We will use this below.

Let \( A = \{z \mid |\text{Re} z| < \pi/2\} \) and \( B = \mathbb{C} \setminus \{z \mid \text{Im} z = 0 \text{ and } |z| \geq 1\} \). We are asked to show that \( z \mapsto \sin z \) carries \( A \) onto \( B \). Since \( \sin(\overline{z}) = \overline{\sin z} \) and \( \sin(-z) = -\sin(z) \), it is sufficient to show that \( z \mapsto \sin z \) maps \( A' = \{z \mid 0 \leq \text{Re} z < \pi/2 \text{ and } \text{Im} z \geq 0\} \) onto \( B' = \{z \mid \text{Re} z \geq 0 \text{ and } \text{Im} z \geq 0\} \setminus \{z \mid \text{Im} z = 0 \text{ and } \text{Re} z \geq 1\} \). We will argue geometrically.

Fix \( x_0 \in [0, \pi/2) \) and consider the vertical ray defined by \( z = x_0 + iy, \, y \geq 0 \). Writing \( \sin z = u + iv \) we find that the real and imaginary parts of the image of this ray under \( \sin z \) satisfy
\[
u = \sin x_0 \cosh y \\
v = \cos x_0 \sinh y.
\]
If \( x_0 = 0 \) then we have, in fact, \( u = 0 \) and \( v = \sinh y \). As \( y \geq 0 \), the points \( u + iv \) then trace out the non-negative imaginary axis. If \( x_0 \neq 0 \) then we know that \( \sin x_0 \neq 0 \) and so as \( y \geq 0 \) varies, we find that the equations expressing \( u \) and \( v \) above give the standard parametrization of the portion of hyperbola
\[
\frac{u^2}{\sin^2 x_0} - \frac{v^2}{\cos^2 x_0} = 1.
\]
that lies in the first quadrant.

As \( x_0 \) increases from 0 to \( \pi/2 \), the rays \( z = x_0 + iy \) cover all of \( A' \). Moreover, the images of our rays start at the positive imaginary axis and proceed to “bend toward” the part of
the real axis for which \( u > 1 \) (sketch a few image curves for \( x_0 \) approaching \( \pi/2 \) so see what I mean). It is then intuitively clear that as \( x_0 \) moves across the interval \([0, \pi/2)\) the image curves fill the set \( B' \), which is what we wanted to show.

A more careful proof could be given by showing that given any \( w \in B' \) there is an \( x_0 \) so that the image of the ray \( z = x_0 + iy \) passes through \( w \). This is tedious (but not difficult) and the details will not be given here.

1.4.2.

(a) For any \( z_1, z_2 \in \mathbb{C} \) we have, by properties of \(| \cdot |\),

\[
|\text{Re} \ z_1 - \text{Re} \ z_2| = |\text{Re} (z_1 - z_2)| \leq |z_1 - z_2|
\]

and

\[
|\text{Im} \ z_1 - \text{Im} \ z_2| = |\text{Im} (z_1 - z_2)| \leq |z_1 - z_2|
\]

and finally (by the triangle inequality)

\[
|z_1 - z_2| = |\text{Re} (z_1 - z_2) + i \text{Im} (z_1 - z_2)|
\]

\[
\leq |\text{Re} (z_1 - z_2)| + |i \text{Im} (z_1 - z_2)|
\]

\[
= |\text{Re} z_1 - \text{Re} z_2| + |\text{Im} z_1 - \text{Im} z_2|.
\]

(b) Theorem: Let \( f(z) = u(x, y) + iv(x, y) \) and \( z_0 = x_0 + iy_0 \). The limit

\[
\lim_{z \to z_0} f(z)
\]

exists if and only if both of the limits

\[
\lim_{(x,y) \to (x_0,y_0)} u(x,y), \quad \lim_{(x,y) \to (x_0,y_0)} v(x,y)
\]

exist. In either case

\[
\lim_{z \to z_0} f(z) = \lim_{(x,y) \to (x_0,y_0)} u(x,y) + i \lim_{(x,y) \to (x_0,y_0)} v(x,y).
\]

Proof: (\( \Rightarrow \)) Suppose \( \lim_{z \to z_0} f(z) = L \in \mathbb{C} \). Let \( \epsilon > 0 \). Choose \( \delta > 0 \) so that \( 0 < |z - z_0| < \delta \) implies \( |f(z) - L| < \epsilon \). Let \( d(P, Q) \) denote the usual distance between two points \( P, Q \in \mathbb{R}^2 \). Then \( d((x, y), (x_0, y_0)) = \sqrt{(x - x_0)^2 + (y - y_0)^2} = |(x - x_0) + i(y - y_0)| = |(x + iy) - z_0| \). Therefore, \( 0 < d((x, y), (x_0, y_0)) < \delta \) implies \( 0 < |(x + iy) - z_0| < \delta \) which in turn implies that

\[
|u(x, y) - \text{Re} L| = |\text{Re} f(x + iy) - \text{Re} L| \leq |f(x + iy) - L| < \epsilon
\]

where we have used part (a) to obtain the first inequality. Since \( \epsilon > 0 \) was arbitrary we conclude that

\[
\lim_{(x,y) \to (x_0,y_0)} u(x,y) = \text{Re} L
\]
proving that the limit exists. Replacing Re with Im and \( u(x, y) \) with \( v(x, y) \) in the above argument, we find that we also have
\[
\lim_{(x, y) \to (x_0, y_0)} v(x, y) = \Im L.
\]
This shows that the limits in question exist and also that
\[
\lim_{z \to z_0} f(z) = L = \Re L + i \Im L = \lim_{(x, y) \to (x_0, y_0)} u(x, y) + i \lim_{(x, y) \to (x_0, y_0)} v(x, y).
\]

(\( \Leftarrow \)) Now suppose that
\[
\lim_{(x, y) \to (x_0, y_0)} u(x, y) = a, \quad \lim_{(x, y) \to (x_0, y_0)} v(x, y) = b.
\]
Let \( \epsilon > 0 \). As above, we let \( d(P, Q) \) denote the distance between two points \( P, Q \in \mathbb{R}^2 \).
We can choose a \( \delta_1 > 0 \) so that \( 0 < d((x, y), (x_0, y_0)) < \delta_1 \) implies \( |u(x, y) - a| < \epsilon/2 \). Likewise, we can choose a \( \delta_2 > 0 \) so that \( 0 < d((x, y), (x_0, y_0)) < \delta_2 \) implies \( |v(x, y) - b| < \epsilon/2 \). Let \( \delta = \min\{\delta_1, \delta_2\} > 0 \). Then if \( 0 < |z - z_0| < \delta \) and \( z = x + iy \) we have \( 0 < d((x, y), (x_0, y_0)) < \delta \) and so
\[
|f(z) - (a + bi)| \leq |\Re f(z) - \Re(a + bi)| + |\Im f(z) - \Im(a + bi)|
\]
\[
= |u(x, y) - a| + |v(x, y) - b|
\]
\[
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
Since \( \epsilon > 0 \) was arbitrary we conclude that
\[
\lim_{z \to z_0} f(z) = a + bi = \lim_{(x, y) \to (x_0, y_0)} u(x, y) + i \lim_{(x, y) \to (x_0, y_0)} v(x, y)
\]
which is what we needed to show.

1.4.4. We must prove that \( \mathbb{C} \setminus \{z_0\} \) is open. Let \( z_1 \in \mathbb{C} \setminus \{z_0\} \). Then \( z_1 \neq z_0 \) and so \( r = |z_1 - z_0|/2 > 0 \). Let \( z \in D(z_1, r) \). Then by the reverse triangle inequality
\[
|z - z_0| = |z_1 - z_0 + z - z_1| \geq |z_1 - z_0| - |z - z_1| = 2r - |z - z_1| > 2r - r = r > 0
\]
so that \( z \neq z_0 \) and hence \( z \in \mathbb{C} \setminus \{z_0\} \). Since \( z \in D(z_1, r) \) was arbitrary this proves that \( D(z_1, r) \subset \mathbb{C} \setminus \{z_0\} \), and since \( z_1 \in \mathbb{C} \setminus \{z_0\} \) was arbitrary this proves that \( \mathbb{C} \setminus \{z_0\} \) is open. Hence \( \{z_0\} \) is closed.

1.4.8. Let \( z_0 \in \mathbb{C} \). Let \( \epsilon > 0 \) and set \( \delta = \epsilon > 0 \). Then if \( |z - z_0| < \delta \) the reverse triangle inequality implies
\[
||z| - |z_0|| \leq |z - z_0| < \delta = \epsilon.
\]
Since \( \epsilon > 0 \) was arbitrary this proves that \( |z| \) is continuous at the point \( z_0 \). Since \( z_0 \in \mathbb{C} \) was arbitrary we conclude that \( |z| \) is continuous on \( \mathbb{C} \).

1.4.10. Let \( \epsilon > 0 \). Since \( \lim_{w \to a} h(w) = c \) there is a \( \delta_1 > 0 \) so that \( 0 < |w - a| < \delta_1 \) implies \( |h(w) - c| < \epsilon \). Since \( \lim_{z \to z_0} f(z) = a \) and \( \delta_1 > 0 \) there is a \( \delta > 0 \) so that \( 0 < |z - z_0| < \delta \)
implies that $|f(z) - a| < \delta_1$. Therefore $0 < |z - z_0| < \delta$ implies that $|f(z) - a| < \delta_1$ which implies that (taking $w = f(z)$) $|h(f(z)) - c| < \epsilon$. Since $\epsilon > 0$ was arbitrary this proves that

$$\lim_{z \to z_0} h(f(z)) = c$$

which is what we sought to prove.

1.4.14.

(a) The set $\{z \mid \text{Im} z > 2\}$ is open but not closed.
(b) The set $\{z \mid 1 \leq |z| \leq 2\}$ is closed but not open.
(c) The set $\{z \mid -1 < \text{Re} z \leq 2\}$ is neither open nor closed.

1.4.16(i).

(a) The set $\{z \mid 1 < \text{Re} z \leq 2\}$ is path connected and hence connected.
(b) The set $\{z \mid 2 \leq |z| \leq 3\}$ is path connected and hence connected.
(c) The set $\{z \mid |z| \leq 5 \text{ and } |\text{Im} z| \geq 1\}$ is not connected.

1.4.20. Let $A_1, \ldots, A_n \subset \mathbb{C}$ be open sets. Let $z \in \bigcap_{i=1}^n A_i$. Then for each $i$ we know that $z \in A_i$ and therefore we may choose $r_i > 0$ so that $D(z, r_i) \subset A_i$. Let $r = \min_{1 \leq i \leq n} r_i$. Then $r > 0$ and for any $i$ we have $r \leq r_i$ so that $D(z, r) \subset D(z, r_i) \subset A_i$. Hence $D(z, r) \subset \bigcap_{i=1}^n A_i$. Since $z$ was an arbitrary element of the intersection we conclude that $\bigcap_{i=1}^n A_i$ is open.

1.4.22. Let $z_0 \in A$. Since $f$ is continuous on $A$ we have $\lim_{z \to z_0} f(z) = f(z_0)$. Since $h$ is continuous on $f(A)$ it is, in particular, continuous at $f(z_0)$. Therefore, if we take $a = f(z_0)$ in part (i) of the proposition we have

$$\lim_{z \to z_0} h(f(z)) = h(f(z_0))$$

which proves that $h(f(z))$ is continuous at $z_0$. Since $z_0 \in A$ was arbitrary this proves that $h(f(z))$ is continuous on $A$. 

4