1.3.10. *Claim:* Using the branch of the square root function given in the problem, $\sqrt{z^2} = z$ iff $z = 0$ or $z = re^{i\theta}$ with $r > 0$ and $0 \le \theta < \pi$.

Proof: ($\Leftarrow$) Suppose $z = re^{i\theta}$ with $r \ge 0$ and $0 \le \theta < \pi$. Then $z^2 = r^2 e^{i2\theta}$ and $0 \le 2\theta < 2\pi$ so that

$$\sqrt{z^2} = (r^2)^{1/2} e^{i2\theta/2} = re^{i\theta} = z.$$

($\Rightarrow$) Suppose z is *not* of the desired form. We will show that $\sqrt{z^2} \ne z$. In this case $z = re^{i\theta}$ with $r > 0$ and $\pi \le \theta < 2\pi$. Then $z^2 = r^2 e^{i2\theta} = r^2 e^{i(2\theta - 2\pi)}$ and since $0 \le 2\theta - 2\pi < 2\pi$ we have

$$\sqrt{z^2} = (r^2)^{1/2} e^{i(2\theta-2\pi)/2} = re^{i\theta} e^{i\pi} = -z.$$

Since $z \ne 0$, this is *not* the same as z.

1.3.34. If $z = x + iy$, it is shown on page 38 of our text that

$$\sin z = \sin x \cosh y + i \cos x \sinh y.$$

We will use this below.

Let $A = \{z \,|\, |\operatorname{Re} z| < \pi/2\}$ and $B = \mathbb{C} \setminus \{z \,|\, \operatorname{Im} z = 0 \text{ and } |z| \ge 1\}$. We are asked to show that $z \mapsto \sin z$ carries A onto B. Since $\sin(\overline{z}) = \overline{\sin z}$ and $\sin(-z) = -\sin(z)$, it is sufficient to show that $z \mapsto \sin z$ maps $A' = \{z \,|\, 0 \le \operatorname{Re} z < \pi/2 \text{ and } \operatorname{Im} z \ge 0\}$ onto $B' = \{z \,|\, \operatorname{Re} z \ge 0 \text{ and } \operatorname{Im} z \ge 0\} \setminus \{z \,|\, \operatorname{Im} z = 0 \text{ and } \operatorname{Re} z \ge 1\}$. We will argue geometrically.

Fix $x_0 \in [0, \pi/2)$ and consider the vertical ray defined by $z = x_0 + iy$, $y \ge 0$. Writing $\sin z = u + iv$ we find that the real and imaginary parts of the image of this ray under $\sin z$ satisfy

$$\begin{aligned} u &= \sin x_0 \cosh y \\ v &= \cos x_0 \sinh y. \end{aligned}$$

If $x_0 = 0$ then we have, in fact, $u = 0$ and $v = \sinh y$. As $y \ge 0$, the points $u + iv$ then trace out the non-negative imaginary axis. If $x_0 \ne 0$ then we know that $\sin x_0 \ne 0$ and so as $y \ge 0$ varies, we find that the equations expressing u and v above give the standard parametrization of the portion of hyperbola

$$\frac{u^2}{\sin^2 x_0} - \frac{v^2}{\cos^2 x_0} = 1.$$

that lies in the first quadrant.

As x_0 increases from 0 to $\pi/2$, the rays $z = x_0 + iy$ cover all of A'. Moreover, the images of our rays start at the positive imaginary axis and proceed to "bend toward" the part of the real axis for which $u > 1$ (sketch a few image curves for x_0 approaching $\pi/2$ so see what I mean). It is then intuitively clear that as x_0 moves across the interval $[0, \pi/2)$ the image curves fill the set B', which is what we wanted to show.

A more careful proof could be given by showing that given any $w \in B'$ there is an x_0 so that the image of the ray $z = x_0 + iy$ passes through w. This is tedious (but not difficult) and the details will not be given here.

1.4.2.

(a) For any $z_1, z_2 \in \mathbb{C}$ we have, by properties of $|\cdot|$,

$$|\operatorname{Re} z_1 - \operatorname{Re} z_2| = |\operatorname{Re}(z_1 - z_2)| \leq |z_1 - z_2|$$

and

$$|\operatorname{Im} z_1 - \operatorname{Im} z_2| = |\operatorname{Im}(z_1 - z_2)| \leq |z_1 - z_2|$$

and finally (by the triangle inequality)

$$\begin{aligned} |z_1 - z_2| &= |\operatorname{Re}(z_1 - z_2) + i \operatorname{Im}(z_1 - z_2)| \\ &\leq |\operatorname{Re}(z_1 - z_2)| + |i \operatorname{Im}(z_1 - z_2)| \\ &= |\operatorname{Re} z_1 - \operatorname{Re} z_2| + |\operatorname{Im} z_1 - \operatorname{Im} z_2|. \end{aligned}$$

(b) *Theorem:* Let $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$. The limit

$$\lim_{z \to z_0} f(z)$$

exists if and only if both of the limits

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y), \qquad \lim_{(x,y) \to (x_0,y_0)} v(x, y)$$

exist. In either case

$$\lim_{z \to z_0} f(z) = \lim_{(x,y) \to (x_0,y_0)} u(x, y) + i \lim_{(x,y) \to (x_0,y_0)} v(x, y).$$

Proof: ($\Rightarrow$) Suppose $\lim_{z \to z_0} f(z) = L \in \mathbb{C}$. Let $\epsilon > 0$. Choose $\delta > 0$ so that $0 < |z - z_0| < \delta$ implies $|f(z) - L| < \epsilon$. Let $d(P, Q)$ denote the usual distance between two points $P, Q \in \mathbb{R}^2$. Then $d((x, y), (x_0, y_0)) = \sqrt{(x - x_0)^2 + (y - y_0)^2} = |(x - x_0) + i(y - y_0)| = |(x + iy) - z_0|$. Therefore, $0 < d((x, y), (x_0, y_0)) < \delta$ implies $0 < |(x + iy) - z_0| < \delta$ which in turn implies that

$$|u(x, y) - \operatorname{Re} L| = |\operatorname{Re} f(x + iy) - \operatorname{Re} L| \leq |f(x + iy) - L| < \epsilon$$

where we have used part (a) to obtain the first inequality. Since $\epsilon > 0$ was arbitrary we conclude that

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = \operatorname{Re} L$$

proving that the limit exists. Replacing Re with Im and $u(x,y)$ with $v(x,y)$ in the above argument, we find that we also have

$$\lim_{(x,y)\to(x_0,y_0)} v(x,y) = \operatorname{Im} L.$$

This shows that the limits in question exist and also that

$$\lim_{z\to z_0} f(z) = L = \operatorname{Re} L + i \operatorname{Im} L = \lim_{(x,y)\to(x_0,y_0)} u(x,y) + i \lim_{(x,y)\to(x_0,y_0)} v(x,y).$$

($\Leftarrow$) Now suppose that

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = a, \qquad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = b.$$

Let $\epsilon > 0$. As above, we let $d(P,Q)$ denote the distance between two points $P,Q \in \mathbb{R}^2$. We can choose a $\delta_1 > 0$ so that $0 < d((x,y),(x_0,y_0)) < \delta_1$ implies $|u(x,y) - a| < \epsilon/2$. Likewise, we can choose a $\delta_2 > 0$ so that $0 < d((x,y),(x_0,y_0)) < \delta_2$ implies $|v(x,y) - b| < \epsilon/2$. Let $\delta = \min\{\delta_1,\delta_2\} > 0$. Then if $0 < |z - z_0| < \delta$ and $z = x + iy$ we have $0 < d((x,y),(x_0,y_0)) < \delta$ and so

$$\begin{aligned} |f(z) - (a+bi)| &\leq |\operatorname{Re} f(z) - \operatorname{Re}(a+bi)| + |\operatorname{Im} f(z) - \operatorname{Im}(a+bi)| \\ &= |u(x,y) - a| + |v(x,y) - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary we conclude that

$$\lim_{z\to z_0} f(z) = a + bi = \lim_{(x,y)\to(x_0,y_0)} u(x,y) + i \lim_{(x,y)\to(x_0,y_0)} v(x,y)$$

which is what we needed to show.

1.4.4. We must prove that $\mathbb{C} \setminus \{z_0\}$ is open. Let $z_1 \in \mathbb{C} \setminus \{z_0\}$. Then $z_1 \neq z_0$ and so $r = |z_1 - z_0|/2 > 0$. Let $z \in D(z_1, r)$. Then by the reverse triangle inequality

$$|z - z_0| = |z_1 - z_0 + z - z_1| \geq |z_1 - z_0| - |z - z_1| = 2r - |z - z_1| > 2r - r = r > 0$$

so that $z \neq z_0$ and hence $z \in \mathbb{C} \setminus \{z_0\}$. Since $z \in D(z_1, r)$ was arbitrary this proves that $D(z_1, r) \subset \mathbb{C} \setminus \{z_0\}$, and since $z_1 \in \mathbb{C} \setminus \{z_0\}$ was arbitrary this proves that $\mathbb{C} \setminus \{z_0\}$ is open. Hence $\{z_0\}$ is closed.

1.4.8. Let $z_0 \in \mathbb{C}$. Let $\epsilon > 0$ and set $\delta = \epsilon > 0$. Then if $|z - z_0| < \delta$ the reverse triangle inequality implies

$$||z| - |z_0|| \leq |z - z_0| < \delta = \epsilon.$$

Since $\epsilon > 0$ was arbitrary this proves that $|z|$ is continuous at the point z_0. Since $z_0 \in \mathbb{C}$ was arbitrary we conclude that $|z|$ is continuous on $\mathbb{C}$.

1.4.10. Let $\epsilon > 0$. Since $\lim_{w\to a} h(w) = c$ there is a $\delta_1 > 0$ so that $0 < |w - a| < \delta_1$ implies $|h(w) - c| < \epsilon$. Since $\lim_{z\to z_0} f(z) = a$ and $\delta_1 > 0$ there is a $\delta > 0$ so that $0 < |z - z_0| < \delta$

implies that $|f(z) - a| < \delta_1$. Therefore $0 < |z - z_0| < \delta$ implies that $|f(z) - a| < \delta_1$ which implies that (taking $w = f(z)$) $|h(f(z)) - c| < \epsilon$. Since $\epsilon > 0$ was arbitrary this proves that

$$\lim_{z \to z_0} h(f(z)) = c$$

which is what we sought to prove.

1.4.14.

(a) The set $\{z \mid \operatorname{Im} z > 2\}$ is open but not closed.

(b) The set $\{z \mid 1 \le |z| \le 2\}$ is closed but not open.

(c) The set $\{z \mid -1 < \operatorname{Re} z \le 2\}$ is neither open nor closed.

1.4.16(i).

(a) The set $\{z \mid 1 < \operatorname{Re} z \le 2\}$ is path connected and hence connected.

(b) The set $\{z \mid 2 \le |z| \le 3\}$ is path connected and hence connected.

(c) The set $\{z \mid |z| \le 5 \text{ and } |\operatorname{Im} z| \ge 1\}$ is not connected.

1.4.20. Let $A_1, \ldots, A_n \subset \mathbb{C}$ be open sets. Let $z \in \bigcap_{i=1}^n A_i$. Then for each i we know that $z \in A_i$ and therefore we may choose $r_i > 0$ so that $D(z, r_i) \subset A_i$. Let $r = \min_{1 \le i \le n} r_i$. Then $r > 0$ and for any i we have $r \le r_i$ so that $D(z, r) \subset D(z, r_i) \subset A_i$. Hence $D(z, r) \subset \bigcap_{i=1}^n A_i$. Since z was an arbitrary element of the intersection we conclude that $\bigcap_{i=1}^n A_i$ is open.

1.4.22. Let $z_0 \in A$. Since f is continuous on A we have $\lim_{z \to z_0} f(z) = f(z_0)$. Since h is continuous on $f(A)$ it is, in particular, continuous at $f(z_0)$. Therefore, if we take $a = f(z_0)$ in part (i) of the proposition we have

$$\lim_{z \to z_0} h(f(z)) = h(f(z_0))$$

which proves that $h(f(z))$ is continuous at z_0. Since $z_0 \in A$ was arbitrary this proves that $h(f(z))$ is continuous on A.