1.5.2.

(a) The function $f(z) = 3z^2+7z+5$ is a polynomial so is analytic everywhere with derivative $f'(z) = 6z + 7$.

(b) The function $f(z) = (2z+3)^4$ is a composition of polynomials so is analytic everywhere with derivative $f'(z) = 8(2z + 3)^3$ (by the chain rule).

(c) The function $f(z) = (3z - 1)/(3 - z)$ is rational and so is analytic where $z \neq 3$. By the quotient rule, its derivative is

$$f'(z) = \frac{8}{(3-z)^2}.$$

1.5.10. Let $f(z) = |z|$ and write $f = u + iv$ and $z = x + iy$. Then $u = \sqrt{x^2 + y^2}$ and $v = 0$. We find that

$$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$
$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

In order for f to be analytic the Cauchy-Riemann equations must hold. That is, we need

$$\frac{x}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} = 0$$

which is impossible, since it requires x and y to be both simultaneously zero and nonzero. Therefore f is differentiable nowhere.

Here's another proof. Since the partial derivatives of $\mathrm{Re}(|z|) = \sqrt{x^2 + y^2}$ do not exist at $(0, 0)$, the Cauchy-Riemann theorem implies that $|z|$ cannot be differentiable at $z = 0$. Suppose that $f(z) = |z|$ were analytic at some $z_0 \in \mathbb{C}$, $z_0 \neq 0$. Then $(f(z))^2 = |z|^2 = z\overline{z}$ would be analytic at z_0 and, since z_0 is nonzero, so too would be $(f(z))^2/z = \overline{z}$. But we know that $g(z) = \overline{z}$ is analytic nowhere, so this is impossible. Therefore $f(z) = |z|$ cannot be analytic at any nonzero complex number either. Hence, $|z|$ is differentiable nowhere, i.e. is not analytic.

1.5.14.

(a) According to Cauchy-Riemann equations, if f is analytic then

$$f' = \frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$$

so that

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right) = \frac{1}{2}(f' + f') = f'.$$

(b) If $f(z) = z = x + iy$ then f is analytic so that by part (a) we have

$$\partial f z = f' = 1.$$

Moreover

$$\frac{\partial f}{\partial x} = 1, \ \frac{\partial f}{\partial y} = i$$

so that

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{f}{y}\right) = \frac{1}{2}(1 - 1) = 0.$$

(c) If $f(z) = \overline{z} = x - iy$ then

$$\frac{\partial f}{\partial x} = 1, \ \frac{\partial f}{\partial y} = -i.$$

Therefore

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right) = \frac{1}{2}(1 - 1) = 0$$

and

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right) = \frac{1}{2}(1 + 1) = 1.$$

(d) We show first that $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ satisfy the sum, product and scalar multiple rules when applied to *complex valued* functions. Let f, g be complex valued functions and let c be a complex scalar. Write $f = s + it$, $g = u + iv$, $c = a + ib$.

Scalar multiplication. We have $cf = (as - bt) + i(at + bs)$ so that by definition and the linearity of derivative of real functions we have

$$\begin{aligned}
\frac{\partial (cf)}{\partial x} &= \frac{\partial (as - bt)}{\partial x} + i\frac{\partial (at + bs)}{\partial x} \\
&= a\frac{\partial s}{\partial x} - b\frac{\partial t}{\partial x} + i\left(a\frac{\partial t}{\partial x} + b\frac{\partial s}{\partial x}\right) \\
&= (a + ib)\left(\frac{\partial s}{\partial x} + i\frac{\partial t}{\partial x}\right) \\
&= c\frac{\partial f}{\partial x}
\end{aligned}$$

which proves the scalar multiplication rule for $\frac{\partial}{\partial x}$. The proof for $\frac{\partial}{\partial y}$ is identical, replacing x by y throughout.

Sum. We have $f+g=(s+u)+i(t+v)$ so that by definition and the linearity of the derivative of real functions we have

$$\begin{aligned}\frac{\partial(f+g)}{\partial x} &= \frac{\partial(s+u)}{\partial x}+i\frac{\partial(t+v)}{\partial x}\\ &= \frac{\partial s}{\partial x}+\frac{\partial u}{\partial x}+i\left(\frac{\partial t}{\partial x}+\frac{\partial v}{\partial x}\right)\\ &= \left(\frac{\partial s}{\partial x}+i\frac{\partial t}{\partial x}\right)+\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\\ &= \frac{\partial f}{\partial x}+\frac{\partial g}{\partial x}\end{aligned}$$

which proves the addition rule for $\frac{\partial}{\partial x}$. The proof for $\frac{\partial}{\partial y}$ is identical, replacing x with y throughout.

Product. We have $fg=(su-tv)+i(tu+sv)$ so that by definition and the product rule for derivatives of real functions we have

$$\begin{aligned}\frac{\partial(fg)}{\partial x} &= \frac{\partial(su-tv)}{\partial x}+i\frac{\partial(tu+sv)}{\partial x}\\ &= s\frac{\partial u}{\partial x}+u\frac{\partial s}{\partial x}-t\frac{\partial v}{\partial x}-v\frac{\partial t}{\partial x}+i\left(t\frac{\partial u}{\partial x}+u\frac{\partial t}{\partial x}+s\frac{\partial v}{\partial x}+v\frac{\partial s}{\partial x}\right)\\ &= \left(\left(s\frac{\partial u}{\partial x}-t\frac{\partial v}{\partial x}\right)+i\left(t\frac{\partial u}{\partial x}+s\frac{\partial v}{\partial x}\right)\right)+\left(\left(u\frac{\partial s}{\partial x}-v\frac{\partial t}{\partial x}\right)+i\left(u\frac{\partial t}{\partial x}+v\frac{\partial s}{\partial x}\right)\right)\\ &= (s+it)\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)+(u+iv)\left(\frac{\partial s}{\partial x}+i\frac{\partial t}{\partial x}\right)\\ &= f\frac{\partial g}{\partial x}+g\frac{\partial f}{\partial x}\end{aligned}$$

which proves the product rule for $\frac{\partial}{\partial x}$. The proof for $\frac{\partial}{\partial y}$ is identical, replacing x with y throughout.

Finally, we are in a position to verify that the sum, product and scalar multiplication rules hold for $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$. We prove a slightly more general statement that includes both of these statements as special cases.

Let $a, b \in \mathbb{C}$ and set $D = c\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}$. I claim that D satisfies the sum, product and scalar multiple rules. Let f, g be complex functions and let $c \in \mathbb{C}$. By what we have already shown we have

$$\begin{aligned}D(cf) &= a\frac{\partial(cf)}{\partial x}+b\frac{\partial(cf)}{\partial y}\\ &= ac\frac{\partial f}{\partial x}+bc\frac{\partial f}{\partial y}\\ &= cD(f)\end{aligned}$$

which proves the scalar multiple rule. Also

$$\begin{aligned} D(f+g) &= a\frac{\partial(f+g)}{\partial x} + b\frac{\partial(f+g)}{\partial y} \\ &= a\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right) + b\left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\right) \\ &= a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + a\frac{\partial g}{\partial x} + b\frac{\partial g}{\partial y} \\ &= D(f) + D(g), \end{aligned}$$

proving the sum rule. Finally

$$\begin{aligned} D(fg) &= a\frac{\partial(fg)}{\partial x} + b\frac{\partial(fg)}{\partial y} \\ &= a\left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right) + b\left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right) \\ &= f\left(a\frac{\partial g}{\partial x} + b\frac{\partial g}{\partial y}\right) + g\left(a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}\right) \\ &= fD(g) + gD(f), \end{aligned}$$

which establishes the product rule for D.

Now we are finished since

$$\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y}$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}$$

are both of the form dealt with above.

(e) ($\Leftarrow$) This is the easy direction. If $a_{nm} = 0$ whenever $m \neq 0$ we have

$$\sum_{n=0}^{N}\sum_{m=0}^{M} a_{nm} z^n \overline{z}^m = \sum_{n=0}^{N} a_{n0} z^n \overline{z}^0 = \sum_{n=0}^{N} a_{n0} z^n$$

which is a polynomial in z and hence analytic everywhere.

($\Rightarrow$) For this we use the result of exercise 13, which says that if $f(z)$ is analytic then

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

Since $\frac{\partial}{\partial \overline{z}}$ obeys the product rule and $\frac{\partial \overline{z}}{\partial \overline{z}} = 1$ an easy induction shows that $\frac{\partial \overline{z}^m}{\partial \overline{z}} = m\overline{z}^{m-1}$ for $m \in \mathbb{Z}^+$. Moreover, since z^n is analytic we know that $\frac{\partial z^n}{\partial \overline{z}} = 0$. Therefore, if the expression in question is analytic we have (by the sum, product and scalar multiple rules)

$$\begin{aligned}
0 &= \frac{\partial}{\partial \overline{z}}\left(\sum_{n=0}^{N}\sum_{m=0}^{M} a_{nm} z^n \overline{z}^m\right) \\
&= \sum_{n=0}^{N}\sum_{m=0}^{M} \frac{\partial}{\partial \overline{z}}\left(a_{nm} z^n \overline{z}^m\right) \\
&= \sum_{n=0}^{N}\sum_{m=0}^{M} a_{nm}\left(z^n \frac{\partial \overline{z}^m}{\partial \overline{z}} + \overline{z}^m \frac{\partial z^n}{\partial \overline{z}}\right) \\
&= \sum_{n=0}^{N}\sum_{m=0}^{M} a_{nm} m z^n \overline{z}^{m-1} \overline{z}^m.
\end{aligned}$$

But the only way that a polynomial can be identically zero is if all of its coefficients vanish. That is, we need $ma_{nm} = 0$ for all m and n. In particular, if $m \neq 0$ we must have $a_{nm} = 0$, which is exactly what we sought to prove.

1.5.16. Before we begin, notice that the statement $au(x, y) + bv(x, y) = c$ with not all of a, b, c equal to zero is equivalent to saying that the values of $f(z) = u(x, y) + iv(x, y)$ lie on a straight line.

(a) We write $u = u(x, y)$, $v = v(x, y)$ etc. to simplify notation. If we apply $\partial/\partial x$ and $\partial/\partial y$ to the equation $au + bv = c$ we obtain

$$\begin{aligned}
a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial x} &= 0 \\
a\frac{\partial u}{\partial y} + b\frac{\partial v}{\partial y} &= 0.
\end{aligned}$$

Since f is analytic we can apply the Cauchy-Riemann equations in the second equality above to obtain the system

$$\begin{aligned}
a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial x} &= 0 \\
b\frac{\partial u}{\partial x} - a\frac{\partial v}{\partial x} &= 0
\end{aligned}$$

or, in matrix form,

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{matrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{matrix} = \begin{matrix} 0 \\ 0 \end{matrix}.$$

The determinant of the coefficient matrix is $-(a^b + b^2)$ which cannot be zero because a, b, c are real and not all zero. Hence, the only solution to the system is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0.$$

Therefore
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$$
and, since A is connected, we conclude that f is constant.

(b) If a, b, c are complex, then the proof above does not apply. However, if we write $a = a_1 + ia_2$, $b = b_1 + ib_2$ and $c = c_1 + ic_2$ with $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$, then the equation $au + bv = c$ is equivalent to the pair of equations
$$\begin{aligned} a_1u + b_1v &= c_1 \\ a_2u + b_2v &= c_2. \end{aligned}$$
Since not all of a, b, c are zero, it must be that in at least one of these equations not all of the constants are zero. We may then apply part (a) to that equation to conclude that f is constant. In other words, the statement is valid for complex a, b, c as well.

1.5.20. Let $A \subset \mathbb{C}$ be connected and open and let $f : A \to \mathbb{C}$ be an analytic function. We are asked to prove the following: if $f(n+1) = 0$ for some $n \in \mathbb{Z}_0^+$ then f is a polynomial of degree $\leq n$. We induct on n.

When $n = 0$ the statement we need to prove is: if $f' = 0$ then f is constant. This is Proposition 1.5.5 and was proven in class. We now assume the statement holds for some $n \geq 0$. Suppose that $f^{(n+2)} = 0$. Then $(f')^{(n+1)} = f^{(n+2)} = 0$ so that, by our assumption, f' is a polynomial of degree $\leq n$. Write
$$f'(z) = \sum_{k=0}^{n} a_k z^k$$
with all $a_k \in \mathbb{C}$. Let
$$h(z) = f(z) - \sum_{k=0}^{n} \frac{a_k}{k+1} z^{k+1}.$$
Then $h(z)$ is analytic on A and
$$h'(z) = f'(z) - \sum_{k=0}^{n} a_k z^k = 0.$$
But we have already seen that this implies h' is constant. If $h' = c \in \mathbb{C}$ then we see that
$$f(z) = h(z) + \sum_{k=0}^{n} \frac{a_k}{k+1} z^{k+1} = c + \sum_{k=0}^{n} \frac{a_k}{k+1} z^{k+1},$$
i.e. f is a polynomial of degree $\leq n+1$. Therefore, if the statement "$f(n+1) = 0$ implies f is a polynomial of degree $\leq n$" holds for any $n \geq 0$ it also holds for $n+1$. By induction, we conclude that the statement is true for all $n \geq 0$.