1.5.22 If $z = x + iy$ then

$$z^4 = (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)$$

so that $u = \mathrm{Re}(z^4) = x^4 - 6x^2y^2 + y^4$ and $v = \mathrm{Im}(z^4) = 4x^3y - 4xy^3$. We find that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(4x^3 - 12xy^2\right) = 12x^2 - 12y^2$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-12x^2y + 4y^3\right) = -12x^2 + 12y^2$$

so that

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(12x^2 - 12y^2\right) + \left(-12x^2 + 12y^2\right) = 0.$$

Since u is a polynomial in x and y its second-order partials are continuous and $\nabla^2 u = 0$, u is harmonic. We leave the analogous computation involving v to the student.

1.5.32 Claim: If u and v are functions defined on an open set A, u and v satisfy the Cauchy-Riemann equations and u is harmonic on A, then v is a harmonic conjugate of u on A.

We are given

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned}$$

Since u and v already satisfy the Cauchy-Riemann equations, to show that v is conjugate to u it suffices to prove that v is also harmonic. Since u is harmonic, it's second-order partial derivatives are all continuous (by definition). The Cauchy-Riemann equations above then imply the same is true of v's second-order partials. So all we need to do to prove that v is harmonic is verify that $\nabla^2 v = 0$. Appealing to the Cauchy-Riemann equations again we have

$$
\begin{aligned}
\nabla^2 v &= \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial y^2} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \\
&= -\frac{\partial^2 u}{\partial x \partial y} + -\frac{\partial^2 u}{\partial y \partial x} \\
&= 0
\end{aligned}
$$

since the continuity of the second-order mixed partials implies their equality. Hence, v is harmonic. As noted above, this completes the proof.

Notice that the only facts we used were that u and v were related by the Cauchy-Riemann equations and that u had continuous second-order partial derivatives. In fact, if these are the only assumptions that we make then we can, in fact, prove that *both* u and v are harmonic and that they are conjugate. □

Now on to the problem at hand. Since we are given that u is harmonic on the disk the claim shows that it is enough to verify that u and v satisfy the Cauchy-Riemann equations. To (hopefully) clarify things slightly, let me alter the notation of the problem. Let's define:

$$v(x,y) = c + \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)\,dt - \int_{x_0}^{x} \frac{\partial u}{\partial y}(s,y_0)\,ds$$

By the Fundamental Theorem of Calculus (the real-variable version) we have

$$
\begin{aligned}
\frac{\partial v}{\partial y}(x,y) &= \frac{\partial}{\partial y}\int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)\,dt - \frac{\partial}{\partial y}\int_{x_0}^{x} \frac{\partial u}{\partial y}(s,y_0)\,ds \\
&= \frac{\partial u}{\partial x}(x,y)
\end{aligned}
$$

since the second expression is not a function of y. This is one of the Cauchy-Riemann equations. Similar reasoning gives

$$\frac{\partial v}{\partial x}(x,y) = \frac{\partial}{\partial x}\int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)\,dt - \frac{\partial u}{\partial y}(x,y_0).$$

Since $\frac{\partial u}{\partial x}$ is continuous we can pass the partial differentiation under the integral sign[1] to obtain

$$\frac{\partial v}{\partial x}(x,y) = \int_{y_0}^{y} \frac{\partial^2 u}{\partial x^2}(x,t)\,dt - \frac{\partial u}{\partial y}(x,y_0).$$

[1] If you'd like to see a proof of this fact let me know and I can provide you with the details.

Since u is harmonic, $\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$. Substituting this into the above and again using the Fundamental Theorem of Calculus we have

$$\begin{aligned}
\frac{\partial v}{\partial x}(x,y) &= \int_{y_0}^{y} \frac{\partial^2 u}{\partial x^2}(x,t)\,dt - \frac{\partial u}{\partial y}(x,y_0) \\
&= -\int_{y_0}^{y} \frac{\partial^2 u}{\partial y^2}(x,t)\,dt - \frac{\partial u}{\partial y}(x,y_0) \\
&= -\left(\frac{\partial u}{\partial y}(x,t)\Big|_{t=y_0}^{t=y} \right) - \frac{\partial u}{\partial y}(x,y_0) \\
&= -\frac{\partial u}{\partial y}(x,y) + \frac{\partial u}{\partial y}(x,y_0) - \frac{\partial u}{\partial y}(x,y_0) \\
&= -\frac{\partial u}{\partial y}(x,y)
\end{aligned}$$

which is the other half of the Cauchy-Riemann equations.

1.6.2

(a) Choose a branch of $\log w$ that is analytic on the real axis and define $3^z = e^{z\log 3}$. Then 3^z is analytic where the branch of log is and

$$\frac{d}{dz}3^z = (\log 3)3^z$$

there.

(b) Choose a ray R emanating from the origin in the complex plane and let $\log w$ denote any branch of the logarithm that is analytic on $\mathbb{C} \setminus R$. Denote by the set $R-1$ the ray translated to the left one unit in the plane (so that it emanates from -1). Then $\log(z+1)$ is analytic on $\mathbb{C} \setminus (R-1)$ and

$$\frac{d}{dz}\log(1+z) = \frac{1}{z+1}$$

there.

(c) Choose a ray R emanating from the origin in the complex plane and let $\log z$ denote any branch of the logarithm that is analytic on $\mathbb{C} \setminus R$. Then $z^(1+i) = e^{(1+i)\log z}$ is analytic on $\mathbb{C} \setminus R$ and

$$\frac{d}{dz}z^{1+i} = (1+i)z^i = (1+i)e^{i\log z}$$

there

(d) Choose a ray R emanating from the origin in the complex plane and let $\log z$ denote any branch of the logarithm that is analytic on $\mathbb{C} \setminus R$. Then $\sqrt{z} = z^{1/2} = e^{\frac{1}{2}\log z}$ is analytic on $\mathbb{C} \setminus R$ and

$$\frac{d}{dz}\sqrt{z} = \frac{1}{2}z^{-1/2} = \frac{1}{2\sqrt{z}}$$

there.

(e) Choose a ray R emanating from the origin in the complex plane and let $\log z$ denote any branch of the logarithm that is analytic on $\mathbb{C} \setminus R$. Then $\sqrt[3]{z} = z^{1/3} = e^{\frac{1}{3}\log z}$ is analytic on $\mathbb{C} \setminus R$ and

$$\frac{d}{dz}\sqrt[3]{z} = \frac{1}{3}z^{-2/3} = \frac{1}{3z^{2/3}} = \frac{1}{3\left(\sqrt[3]{z}\right)^2}$$

there.

1.6.4

(a) Let $\log z$ denote the principal branch of the logarithm so that $\log 1 = 0$. Then $\log z$ is analytic at $z = 1$ and

$$\lim_{z\to 1}\frac{\log z}{z-1} = \frac{d}{dz}\log z\bigg|_{z=1} = \frac{1}{z}\bigg|_{z=1} = 1.$$

If, however, we chose any other branch of the logarithm that was defined at $z = 1$, then $\log 1 = 2n\pi i$ for some nonzero integer n and this can be used to show that the limit above does not exist in this case.

(b) If $f(z) = \overline{z}$, then

$$\lim_{z\to 1}\frac{\overline{z}-1}{z-1} = \lim_{z\to 1}\frac{f(z)-1}{z-1}$$

which is the definition of $f'(1)$. However, we know that $f(z)$ is not analytic anywhere so that $f'(1)$ does not exist. Hence, the limit in question does not exist.

1.6.8 For simplicity, let's assume that both u and v have continuous first-order partial derivatives. Then the function $f = u + iv$ is analytic on A and so are both $(f(z))^2$ and $e^{f(z)}$. One easily checks that

$$\begin{aligned} \mathrm{Re}(f(z))^2 &= u^2 - v^2 = u_1 \\ \mathrm{Im}(f(z))^2 &= 2uv = v_1 \end{aligned}$$

and

$$\begin{aligned} \operatorname{Re} e^{f(z)} &= e^u \cos v = u_2 \\ \operatorname{Im} e^{f(z)} &= e^u \sin v = v_2. \end{aligned}$$

SInce they are the real and imaginary parts of analytic functions, we find that the pairs u_1, v_1 and u_2, v_2 satisfy the Cauchy-Riemann equations on A.

A comment is in order here. The continuity assumption on the partial derivatives of u and v is not really necessary. One can use the multi-variate chain rule to show that if $g(z)$ is analytic and u and v satisfy the Cauchy-Riemann equations then so, too, do the real and imaginary parts of $g(u+iv)$.

1.6.10

(a) We let $\sqrt{w}$ denote the principal branch of the square root. That is, we choose the branch of the argument with $-\pi \le \arg w < \pi$ and set $\sqrt{w} = e^{\frac{1}{2}\log w}$. With this choice $\sqrt{w}$ is analytic on the set $C \setminus \{u+iv \mid u \le 0,\ v = 0\}$. If we let $R = \{u+iv \mid u \le 0,\ v = 0\}$ and $f(z) = z^3 - 1$ then $\sqrt{z^3-1}$ is analytic on $f^{-1}(\mathbb{C} \setminus R) = \mathbb{C} \setminus f^{-1}(R)$. Our goal is to determine $f^{-1}(R)$.

We have $z \in f^{-1}(R)$ iff $f(z) = z^3 - 1 \in R$. This can happen iff $z^3 \in R + 1 = \{u+iv \mid u \le 1,\ v = 0\} = R \cup [0,1]$. If $x \in R$ then $z^3 = x$ has the solutions $z = \sqrt[3]{|x|}e^{i\theta}$ where $\theta = \pi/3, \pi, 5\pi/3$. It follows that z^3 belongs to the ray R iff z belongs to one of the rays emanating from 0 with angle $\pi/3, \pi$ or $5\pi/3$ relative to the real axis. SImilarly, if $x \in [0,1]$ then $z^3 = x$ implies that z belongs to one of the line segments of length 1 making an angle of $0, 2\pi/3$ or $4\pi/3$ with the real axis. If you sketch these six regions together you will see that $f^{-1}(R)$ can be described as follows: it consists of the rays emanating from 0 with angle $\pi/3, \pi$ or $5\pi/3$ relative to the real axis together with their length 1 extensions across the origin. If we call the set just described S, then $\sqrt{z^3-1}$ is analytic on $\mathbb{C} \setminus S$. Now go draw a picture of this set.

(b) Dealing with this function is much easier. Since $\sin w$ is entire, $\sin\sqrt{z}$ is analytic wherever $\sqrt{z}$ is. So, for example, if $\sqrt{z}$ is a branch analytic on $\mathbb{C} \setminus R$ (where R is any ray emanating from the origin) then $\sin\sqrt{z}$ is analytic on the same set.

2.1.2

(a) This is the hardest of the bunch. The curve γ consists of two line segments which are parameterized by

$$\gamma_1(t) = it,\ t \in [0,1]$$

and

$$\gamma_2(t) = (1-t)i + t(i+2),\ t \in [0,1].$$

We have

$$\begin{aligned} x(\gamma_1(t)) &= \operatorname{Re}\gamma_1(t) = 0 \\ \gamma_1'(t) &= i \end{aligned}$$

and

$$\begin{aligned} x(\gamma_2(t)) &= \operatorname{Re}\gamma_2(t) = 2t \\ \gamma_2'(t) &= -i + i + 2 = 2 \end{aligned}$$

so that

$$\begin{aligned} \int_\gamma x\,dz &= \int_{\gamma_1} x\,dz + \int_{\gamma_2} x\,dz \\ &= \int_0^1 (0)(i)\,dt + \int_0^1 (2t)(2)\,dt \\ &= 4\int_0^1 t\,dt \\ &= 2. \end{aligned}$$

(b) This is the easiest of the bunch. A complex antiderivative for $z^2 + 2z + 3$ is $F(z) = \frac{z^3}{3} + z^2 + 3z$ so that by the fundamental theorem of calculus (for complex path integrals)

$$\int_\gamma z^2 + 2z + 3\,dz = F(2+i) - F(1) = \frac{29}{3} + \frac{32}{3}i - \frac{13}{3} = \frac{16}{3} + \frac{32}{3}i.$$

(c) We can parametrize γ by

$$\gamma(t) = 1 + 2e^{it},\ t \in [0, 2\pi].$$

Since $\gamma'(t) = 2ie^{it}$ we have

$$\int_\gamma \frac{1}{z-1}\,dz = \int_0^{2\pi} \frac{1}{(1+2e^{it}) - 1} 2ie^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i.$$

2.1.4 We begin by noting that

$$\frac{1}{z^2 - 2z} = \frac{1}{z(z-2)} = \frac{-1/2}{z} + \frac{1/2}{z-2}.$$

Consequently

$$\int_\gamma \frac{1}{z^2 - 2z}\,dz = -\frac{1}{2}\int_\gamma \frac{1}{z}\,dz + \frac{1}{2}\int_\gamma \frac{1}{z-2}\,dz.$$

Since γ lies entirely in the first and fourth quadrants, we can choose a branch of $\log z$ that is analytic on γ. Since any such branch has $1/z$ as its derivative, the first integral above is zero by the Fundamental Theorem of Calculus for path integrals. On the other hand, if we parameterize γ by $\gamma(t) = 2 + e^{it}$, $t \in [0, 2\pi]$ then

$$\int_\gamma \frac{1}{z-2}\,dz = \int_0^{2\pi} \frac{1}{(2+e^{it})-2} ie^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i.$$

Combining this with our earlier observations we have

$$\int_\gamma \frac{1}{z^2-2z}\,dz = -\frac{1}{2}(0) + \frac{1}{2}2\pi i = \pi i.$$

2.1.6 For convenience we assume that all paths involved are C^1. For a complete proof, one must break each path in question into C^1 pieces and apply the special cases that we prove below.

(a) If γ is parameterized by $\gamma : [a, b] \to \mathbb{C}$ then, using properties of real definite integrals, we have

$$\begin{aligned}\int_\gamma (c_1 f + c_2 g) &= \int_a^b (c_1 f(\gamma(t)) + c_2 g(\gamma(t)))\gamma'(t)\,dt \\ &= c_1 \int_a^b f(\gamma(t))\gamma'(t)\,dt + c_2 \int_a^b g(\gamma(t))\gamma'(t)\,dt \\ &= c_1 \int_\gamma f + c_2 \int_\gamma g.\end{aligned}$$

(b) If γ is parameterized by $\gamma : [a, b] \to \mathbb{R}$ then $-\gamma$ is given, on the same interval, by $(-\gamma)(t) = \gamma(a + b - t)$. By the chain rule $(-\gamma)'(t) = -\gamma'(a + b - t)$ and therefore

$$\begin{aligned}\int_{-\gamma} f &= \int_a^b f(-\gamma(t))(-\gamma)'(t)\,dt \\ &= -\int_a^b f(\gamma(a+b-t))\gamma'(a+b-t)\,dt.\end{aligned}$$

If we make the (real) change of variable $s = a + b - t$ then $ds = -dt$ and

$$\begin{aligned}-\int_a^b f(\gamma(a+b-t))\gamma'(a+b-t)\,dt &= \int_b^a f(\gamma(s))\gamma'(s)\,ds \\ &= -\int_a^b f(\gamma(s))\gamma'(s)\,ds \\ &= -\int_\gamma f\end{aligned}$$

which is what we wanted to show.

(c) If γ_1 is parameterized by $\gamma_1 : [a,b] \to \mathbb{C}$ and γ_2 by $\gamma_2 : [b,c] \to \mathbb{C}$ then we know that

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t) & , t \in [a,b] \\ \gamma_2(t) & , t \in [b,c] \end{cases}$$

and hence

$$(\gamma_1 + \gamma_2)'(t) = \begin{cases} \gamma_1'(t) & , t \in [a,b] \\ \gamma_2'(t) & , t \in [b,c]. \end{cases}$$

It follows that

$$\begin{aligned} \int_{\gamma_1+\gamma_2} f &= \int_a^c f((\gamma_1+\gamma_2)(t))(\gamma_1+\gamma_2)'(t)\,dt \\ &= \int_a^b f((\gamma_1+\gamma_2)(t))(\gamma_1+\gamma_2)'(t)\,dt + \int_b^c f((\gamma_1+\gamma_2)(t))(\gamma_1+\gamma_2)'(t)\,dt \\ &= \int_a^b f(\gamma_1(t))\gamma_1'(t)\,dt + \int_b^c f(\gamma_2(t))\gamma_2'(t)\,dt \\ &= \int_{\gamma_1} f + \int_{\gamma_2} f. \end{aligned}$$

2.1.8

(a) We can parameterize the line segment from 0 to 1+i by $\gamma(t) = t(1+i)$, $t \in [0,1]$. In this case we find that $\gamma'(t) = 1+i$ and $\overline{\gamma(t)}^2 = t^2(1-i)^2$ so that

$$\begin{aligned} \int_\gamma \overline{z}^2\,dz &= \int_0^1 t^2(1-i)^2(1+i)\,dt \\ &= 2(1-i)\int_0^1 t^2\,dt = \frac{2}{3}(1-i) \end{aligned}$$

(b) Since the path in question consists of two line segments, we must parameterize and integrate over each separately. The first segment is given by $\gamma_1(t) = t$, $t \in [0,1]$. Therefore

$$\int_{\gamma_1} \overline{z}^2\,dz = \int_0^1 t^2\,dt = \frac{1}{3}.$$

The second segment can be parameterized as $\gamma_2(t) = 1 + it$, $t \in [0,1]$ and hence

$$\begin{aligned}\int_{\gamma_2} \overline{z}^2\, dz &= \int_0^1 (1-it)^2 i\, dt \\ &= \int_0^1 2t + i(1-t^2)\, dt \\ &= \left(t^2 + i\left(t - \frac{t^2}{2}\right)\right)\Big|_0^1 \\ &= 1 + \frac{i}{2}.\end{aligned}$$

Finally, we have

$$\int_\gamma \overline{z}^2\, dz = \int_{\gamma_1} \overline{z}^2\, dz + \int_{\gamma_2} \overline{z}^2\, dz = \frac{4}{3} + \frac{i}{2}.$$

If $\overline{z}^2$ were the derivative of an analytic function then, by the Fundamental Theorem, the values of all path integrals of $\overline{z}^2$ beginning at 0 and ending at $1+i$ would be the same. Since the computation above shows that this is not the case, we conclude that $\overline{z}^2$ has no complex antiderivative.

2.1.10 If $|z| = 2$ then by the reverse triangle inequality we have

$$|z^2+1| \geq |z^2| - |1| = |z|^2 - 1 = 2^2 - 1 = 3$$

so that

$$\left|\frac{1}{z^2+1}\right| = \frac{|1|}{|z^2+1|} \leq \frac{1}{3}.$$

Hence, if C is the arc of the circle $|z| = 2$ lying in the first quadrant then

$$\left|\int_C \frac{dz}{2+z^2}\right| \leq \frac{1}{3}\ell(C) = \frac{\pi}{3}.$$

2.1.12 On the set $\mathbb{C}\setminus\{z \mid \operatorname{Re} z \leq 0\}$ the principal branch of the logarithm is an antiderivative of $1/z$. By the Fundamental Theorem of Calculus it follows immediately that

$$\int_\gamma \frac{1}{z}\, dz = 0$$

for any closed curve γ in $\mathbb{C} \setminus \{z \mid \operatorname{Re} z \leq 0\}$.