2.2.2 The function $f(z) = 1/z^2$ is analytic in $\mathbb{C} \setminus \{0\}$ and since γ contains 0, it is homotopic so some number of copies of the unit circle in this set. Therefore the Deformation Theorem implies that

$$\int_\gamma \frac{dz}{z^2} = \int_{|z|=1} \frac{dz}{z^2} = 0$$

by a previous computation.

One can also observe that $F(z) = -1/z$ is an antiderivative for the integrand on the set $\mathbb{C} \setminus \{0\}$ and therefore by the Fundamental Theorem of Calculus

$$\int_\gamma \frac{dz}{z^2} = 0$$

for any closed curve not passing through 0.

2.2.4 Let $\log z$ denote the function defined on the region A in question that satisfies $e^{\log z} = z$ for all $z \in A$. When we originally introduced the complex logarithm we proved that if $e^w = z$ then $w = \log |z| + i \arg z$ for *some* choice of the argument. It follows that the equation $\log z = \log r + i\theta$ is valid at each point $z \in A$ for *some* choice of θ at each point. The observation to be made here is that since $\log z$ is analytic on A it is continuous on A and therefore so, too, is $\theta = \theta(z)$. From this it follows that as we "spiral" around the origin in A in the counterclockwise sense, $\theta(z)$ must continually increase. In particular, given the shape of A, we find that $\theta(z)$ cannot be restricted to lie in an interval of length 2π.

2.2.6 Let $\log z$ denote the principal branch of the logarithm. Then γ is contained in the region where $F(z) = z^2/2 - \log z$ is analytic and $F'(z) = z - 1/z$ there. Hence, the Fundamental Theorem of Calculus gives

$$\int_\gamma \left(z - \frac{1}{z}\right) dz = F(i) - F(1) = -1 - \frac{i\pi}{2}.$$

2.2.8 The function $f(z) = \dfrac{1}{z^3(z^2+10)}$ is analytic on $A = \{1/2 < |z| < 3\}$. It is easy to see that γ_1 is homotopic to γ_2 in A and therefore the Deformation Theorem implies

$$\int_{\gamma_1} \frac{dz}{z^3(z^2+10)} = \int_{\gamma_2} \frac{dz}{z^3(z^2+10)}.$$

2.3.7

(a) The curve γ is an ellipse about the origin traversed once counterclockwise. This curve is homotopic in $\mathbb{C}\backslash\{0\}$ to the unit circle C. Since $1/z$ is analytic in $\mathbb{C}\backslash\{0\}$ the deformation theorem implies that

$$\int_\gamma \frac{dz}{z} = \int_C \frac{dz}{z} = 2\pi i.$$

(b) The function $F(z) = -1/z$ is an antiderivative for $f(z) = 1/z^2$ on the set $\mathbb{C}\setminus\{0\}$. Since γ is a closed curve in this set the Fundamental Theorem of Calculus gives

$$\int_\gamma \frac{dz}{z^2} = 0.$$

(c) The function e^z/z is analytic on the disk $D(2, 3/2)$. Since $\gamma \subset D(2, 3/2)$, Cauchy's Theorem implies

$$\int_\gamma \frac{e^z dz}{z} = 0.$$

(d) We begin by observing that

$$\frac{1}{z^2-1} = \frac{1}{2(z-1)} - \frac{1}{2(z+1)}.$$

Therefore

$$\int_\gamma \frac{dz}{z^2-1} = \int_\gamma \frac{dz}{2(z-1)} - \int_\gamma \frac{dz}{2(z+1)}.$$

The function $1/2(z+1)$ is analytic on $D(1, 2)$ so, since $\gamma \subset D(1, 2)$, Cauchy's Theorem gives

$$\int_\gamma \frac{dz}{2(z+1)} = 0.$$

On the other hand, by previous work we know that

$$\int_\gamma \frac{dz}{2(z-1)} = \frac{1}{2}\int_\gamma \frac{dz}{z-1} = \frac{1}{2}2\pi i = \pi i.$$

Hence

$$\int_\gamma \frac{dz}{z^2-1} = \pi i + 0 = \pi i.$$

2.3.9

(a) Let C denote the unit circle. Since $1/z$ is analytic in $\mathbb{C} \setminus \{0\}$, if γ is homotopic to C in $\mathbb{C} \setminus \{0\}$ then

$$\int_\gamma \frac{dz}{z} = \int_C \frac{dz}{z} = 2\pi i$$

by the Deformation Theorem and an earlier computation.

(b) The curve γ in this case is an ellipse about the origin traversed once counter clockwise, which is homotopic in $\mathbb{C} \setminus \{0\}$ to the unit circle. By part (a)

$$\int_\gamma \frac{dz}{d} = 2\pi i.$$

2.3.10 The function $F(z) = \dfrac{1}{2(1-z)^2}$ is an antiderivative for $f(z) = \dfrac{1}{(1-z)^3}$ on the set $\mathbb{C} \setminus \{1\}$. Since all of the curves in question lie entirely in this set, all of the integrals are zero by the Fundamental Theorem of Calculus.