2.4.2

(a) We have several options at this point, so let's choose the one that involves Cauchy's Integral Formula. First let γ_1 denote the contour that travels from 1 to -1 along the top half of the unit circle and then returns to 1 along the real axis. Let γ_2 denote the contour that travels from -1 to 1 along the bottom half of the unit circle and then returns to -1 along the real axis. We then have $\gamma = \gamma_1 + \gamma_2$ so that

$$\int_\gamma \frac{z^2-1}{z^2+1}\,dz = \int_{\gamma_1} \frac{z^2-1}{z^2+1}\,dz + \int_{\gamma_2} \frac{z^2-1}{z^2+1}\,dz.$$

The function $f(z) = \dfrac{z^2-1}{z+i}$ is analytic inside and on γ_1 so that Cauchy's Integral Formula gives

$$i = f(i) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(z)}{(z-i)}\,dz = \frac{1}{2\pi i}\int_{\gamma_1} \frac{z^2-1}{z^2+1}\,dz$$

or

$$\int_{\gamma_1} \frac{z^2-1}{z^2+1}\,dz = -2\pi.$$

Likewise, with $g(z) = \dfrac{z^2-1}{z-i}$ we have

$$-i = g(i) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{g(z)}{(z-i)}\,dz = \frac{1}{2\pi i}\int_{\gamma_2} \frac{z^2-1}{z^2+1}\,dz$$

so that

$$\int_{\gamma_2} \frac{z^2-1}{z^2+1}\,dz = 2\pi.$$

Thus

$$\int_{\gamma} \frac{z^2-1}{z^2+1}\,dz = -2\pi + 2\pi = 0.$$

(b) This one is much easier. Since $\sin e^z$ is analytic inside and on γ, the Cauchy integral formula gives

$$\sin 1 = \sin e^0 = \frac{1}{2\pi i}\int_\gamma \frac{\sin e^z}{z}\,dz$$

so that

$$\int_\gamma \frac{\sin e^z}{z}\,dz = 2\pi(\sin 1)i.$$

2.4.5

(a) We know the integral must be zero because $1/z^3$ has an antiderivative on $C \setminus \{0\}$, namely $-1/2z^2$, but we can also establish this fact using the Cauchy Integral Formula. In particular, if $f(z) = 1$ for all z then we have

$$0 = f''(0) = \frac{2!}{2\pi i} \int_\gamma \frac{f(z)}{z^3}\, dz = \frac{1}{\pi i} \int_\gamma \frac{dz}{z^3}$$

which gives the result.

(b) If we apply Cauchy's Integral Formula with $g(z) = \sin z$ we find that

$$\frac{3!}{2\pi i} \int_\gamma \frac{\sin z}{z^4}\, dz = \frac{3!}{2\pi i} \int_\gamma \frac{g(z)}{z^4}\, dz = g'''(0) = -\cos 0 = -1.$$

Hence

$$\int_\gamma \frac{\sin z}{z^4}\, dz = -\frac{i\pi}{3}.$$

2.4.6 Since f is analytic on A, the Cauchy Integral Formula gives

$$f'(z_0) I(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^2}\, dz$$

But f' is analytic on A, too, so we also have

$$f'(z_0) I(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{z - z_0}\, dz.$$

It follows that

$$\int_\gamma \frac{f(z)}{(z - z_0)^2}\, dz = \int_\gamma \frac{f'(z)}{z - z_0}\, dz.$$

2.4.8 Fix $z_0 \in \mathbb{C}$. Let $\epsilon > 0$. Since $f(z)/z \to 0$ as $z \to \infty$, there exists an $R_0 > 0$ so that $|z| > R_0$ implies $|f(z)/z| < \epsilon$. Let $R > 0$ be any number satisfying $R > |z_0| + R_0$. Then if $|z - z_0| = R$ we have

$$|z| = |z - z_0 + z_0| \geq |z - z_0| - |z_0| = R - |z_0| > R_0$$

so that

$$|f(z)| < \epsilon |z| = \epsilon |z - z_0 + z_0| \leq \epsilon \left(|z - z_0| + |z_0|\right) = \epsilon \left(R + |z_0|\right).$$

Applying Cauchy's Estimates on the circle $|z - z_0| = R$ we have

$$|f'(z_0)| \le \frac{1}{R}\epsilon\,(R + |z_0|) = \frac{R + |z_0|}{R}\epsilon.$$

Since R was an arbitrary number satisfying $R > |z_0| + R_0$, we may let R tend to infinity in the inequality above to obtain

$$|f'(z_0)| \le \epsilon.$$

But $\epsilon > 0$ was arbitrary, so it must be the case that $f'(z_0) = 0$. Finally, $z_0 \in \mathbb{C}$ was arbitrary, so we find that $f'(z) = 0$ for all $z \in \mathbb{C}$. Therefore f is constant.

2.4.16

(a) Of course not. Morera's Theorem applies to the function $f(z) = 1/z^2$ only on the region $A = \mathbb{C} \setminus \{0\}$, since $f(z)$ fails to be continuous at $z = 0$. Therefore it implies nothing about the analyticity of $f(z)$ at $z = 0$.

(b) Again, of course it doesn't. Liouville's Theorem only applies to *entire* functions, and $f(z) = 1/z^2$ is not entire.

2.5.5 Let $h = f - g$. Then h is continuous on $\mathrm{cl}(A)$ and analytic on A. Since $f = g$ on $\mathrm{bd}(A)$, $h = 0$ there. Since A is bounded, the maximum modulus principle implies that

$$0 \le \sup_{z \in A} |h(z)| = \max_{z \in \mathrm{bd}(A)} |h(z)| = 0.$$

Thus $\sup_{z \in A} |h(z)| = 0$ which implies that $f - g = h = 0$ on A. Hence $f = g$ on A.

2.5.18 Let f be an entire function and suppose that $\operatorname{Im} f(z) \le 0$ for all $z \in \mathbb{C}$. Let $g(z) = e^{-if(z)}$. Then g is entire and $|g(z)| = e^{\operatorname{Im} f(z)} \le e^0 = 1$ for all $z \in \mathbb{C}$. By Liouville's Theorem, g must be constant. It follows that $-if(\mathbb{C}) \subset \{c + 2n\pi i,\ |,\ n \in \mathbb{Z}\}$ for some fixed $c \in \mathbb{Z}$. However, $\mathbb{C}$ is connected and f is continuous, so $-if(\mathbb{C})$ is also connected. Since $\{c + 2n\pi i,\ |,\ n \in \mathbb{Z}\}$ consists of only discrete points, it's only connected subsets consist of single points. Hence $-if(\mathbb{C}) = c + 2n\pi i$ for some fixed $n \in \mathbb{Z}$. Thus, $f(z) = ic - 2n\pi$ for all $z \in \mathbb{C}$, i.e. f is constant.