Complex Analysis
Fall 2007

Homework 8: Solutions

**2.R.1**

(a) Since $\sin z$ is entire, its integral around any closed curve is zero by Cauchy's Theorem.

(b) Since $\sin z$ is entire and $\gamma$ has a winding number of 1 about 0, the Cauchy Integral Formula immediately gives

$$0 = \sin 0 = \frac{1}{2\pi i} \int_\gamma \frac{\sin z}{z}\, dz$$

so the integral is zero.

(c) Since $\sin z$ is entire and $\gamma$ has a winding number of 1 about 0, the Cauchy Integral Formula immediately gives

$$1 = \cos 0 = \frac{d}{dz}(\sin z)\bigg|_z = 0 = \frac{1}{2\pi i} \int_\gamma \frac{\sin z}{z^2}\, dz$$

so the integral is $2\pi i$.

(d) Since $f(z) = \sin e^z$ is entire with $f'(z) = e^z \cos e^z$, the Cauchy Integral Formula immediately gives

$$\cos 1 = e^0 \cos e^0 = \frac{1}{2\pi i} \int_{|z|=1} \frac{\sin e^z}{z^2}\, dz$$

so the integral is $2\pi(\cos 1)i$.

**2.R.3** See the textbook's solution.

**2.R.4**

(a) Since $\deg z^2 P(z) = 2 + \deg P(z) \leq \deg Q(z)$ we find that

$$\lim_{z\to\infty} \frac{z^2 P(z)}{Q(z)} = a$$

for some $a \in \mathbb{C}$ (i.e. we are *not* in the case when the limit is infinite). Choose $R > 0$ so that $|z^2 P(z)/Q(z) - a| < 1$ for $|z| \geq R$. Then for $|z| \geq R$ we have

$$\left|\frac{z^2 P(z)}{Q(z)}\right| = \left|\frac{z^2 P(z)}{Q(z)} - a + a\right| \leq \left|\frac{z^2 P(z)}{Q(z)} - a\right| + |a| < 1 + |a|.$$

If we let $M = 1 + |a|$ and divide the inequality above by $|z|^2$ we find that

$$\left|\frac{P(z)}{Q(z)}\right| \leq \frac{M}{|z|^2} \text{ for } |z| \geq R.$$

(b) **Lemma:** Let $P(z)$ and $Q(z)$ be nonzero polynomials with complex coefficients. If $\deg P(z) + 2 \leq \deg Q(z)$ then

$$\lim_{R\to\infty} \int_{|z|=R} \frac{P(z)}{Q(z)}\, dz = 0.$$

**Proof:** First of all, since $Q(z)$ has only finitely many zeros the integrals in question are all defined for sufficiently large $R$. Choose $R_0$ as in part (a), so that

$$\left|\frac{P(z)}{Q(z)}\right| \leq \frac{M}{|z|^2} \text{ for } |z| \geq R_0.$$

If $R > R_0$ and $|z| = R$ then we have

$$\left|\frac{P(z)}{Q(z)}\right| \leq \frac{M}{R^2} \text{ for}$$

and so

$$\left|\int_{|z|=R} \frac{P(z)}{Q(z)}\, dz\right| \leq \frac{M}{R^2} 2\pi R = \frac{2\pi M}{R}.$$

Letting $R \to \infty$ on the right we obtain the result.

**Proposition:** Let $P(z)$ and $Q(z)$ be nonzero polynomials with complex coefficients. Let $z_1, z_2, \ldots z_k \in C$ denote the (finite number of) zeros of $Q(z)$ and let $R_0 = \max\{|z_1|, |z_2|, \ldots, |z_k|\}$. If $\deg P(z) + 2 \leq \deg Q(z)$ and $\gamma$ denotes the circle centered at zero of radius $r > R_0$ then

$$\int_\gamma \frac{P(z)}{Q(z)}\, dz = 0.$$

**Proof:** We see immediately that $P(z)/Q(z)$ is analytic on $A = \{|z| > R_0\}$. If $R > R_0$ and $\gamma_R$ denotes the circle of radius $R$ centered at 0 then $\gamma$ and $\gamma_R$ are homotopic in $A$. Therefore

$$\int_\gamma \frac{P(z)}{Q(z)}\, dz = \int_{\gamma_R} \frac{P(z)}{Q(z)}\, dz.$$

If we now apply the lemma we find that

$$\begin{aligned}\int_\gamma \frac{P(z)}{Q(z)}\, dz &= \lim_{R\to\infty} \int_\gamma \frac{P(z)}{Q(z)}\, dz \\ &= \lim_{R\to\infty} \int_{\gamma_R} \frac{P(z)}{Q(z)}\, dz = 0.\end{aligned}$$

(c) Since the zeros of $z^2+1$ are $\pm i$, $1/(z^2+1)$ and $\gamma$ satisfy the hypotheses of the proposition proven in part (b). Therefore

$$\int_\gamma \frac{dz}{z^2+1} = 0.$$

**2.R.11** The integral in question is just the parametrized version of the line integral

$$\frac{1}{i}\int_{|z|=1} \frac{e^z}{z^2}\,dz.$$

Since $e^z$ is entire, and its own derivative, the Cauchy Integral Formula yields

$$1 = e^0 = \frac{1}{2\pi i}\int_{|z|=1} \frac{e^z}{z^2}\,dz.$$

Thus

$$\int_0^{2\pi} e^{-i\theta}e^{e^{i\theta}}\,d\theta = \frac{1}{i}\int_{|z|=1} \frac{e^z}{z^2}\,dz = 2\pi.$$

**2.R.16** Let $g(z) = f(z) - z_0$. Then $g$ is analytic inside and on the unit circle. If $|z| = 1$ then $|g(z)| = |f(z) - z_0| < r$ since $f$ maps the unit circle inside the disk $D(z_0, r)$. The maximum modulus principle then implies that for $|z'| \leq 1$ we have

$$|g(z')| \leq \max_{|z|=1} |g(z)| < r$$

(The strict inequality follows from the fact that $|g(z)|$ attains its maximum value on $|z| = 1$, and since $|g(z)| \neq r$ for those $z$, the maximum value must not equal $r$ either.) That is, for $z'$ inside the unit disk we have

$$|f(z') - z_0| = |g(z)| < r$$

which proves that $f$ maps the unit disk into the set $D(z_0, r)$.