3.1.4

(a) Let $z \in \mathbb{C} \setminus \{ni : n \in \mathbb{Z}\}$. Then

$$\lim_{n\to\infty} \left| \frac{1/(n^2+z^2)}{1/n^2} \right| = \lim_{n\to\infty} \left| \frac{n^2}{n^2+z^2} \right| = 1.$$

According to the limit comparison test from calculus, the series

$$\sum_{n=0}^{\infty} \left| \frac{1}{n^2+z^2} \right|$$

converges if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges. Since the latter series is known to converge, the former must as well. That is,

$$\sum_{n=0}^{\infty} \frac{1}{n^2+z^2}$$

converges absolutely for $z \in \mathbb{C} \setminus \{ni : n \in \mathbb{Z}\}$.

(b) Let D be any bounded subset of $\mathbb{C} \setminus \{ni : n \in \mathbb{Z}\}$. Then there is an $C > 0$ so that $|z| < C$ for all $z \in D$. For $z \in D$ we then have

$$|n^2 + z^2| \geq |n^2| - |z^2| = n^2 - |z|^2 > n^2 - C^2.$$

If $n > C$ then the above implies that

$$\left| \frac{1}{n^2+z^2} \right| \leq \frac{1}{n^2 - C^2} = M_n$$

for all $z \in D$. Appealing to the limit comparison test as above, we conclude that $\sum_{n>C} M_n$ converges. Hence, the Weierstrass M-test implies that

$$\sum_{n>C} \frac{1}{n^2+z^2}$$

converges uniformly and absolutely on D. Since neither of these modes of convergence is altered by adding finitely many terms (Why? Convince yourself of this.) we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+z^2}$$

converges uniformly and absolutely on D. Since the functions $1/(n^2z^2)$ are all analytic on D, we conclude that the series is an analytic function on D. Finally, since every point $z \in \mathbb{C} \setminus \{ni : n \in \mathbb{Z}\}$ belongs to some bounded subset, we see that the series represents an analytic function on $z \in \mathbb{C} \setminus \{ni : n \in \mathbb{Z}\}$.

3.1.12 Let $\epsilon > 0$ and $A_\epsilon = \{z \in \mathbb{C} : |z| \geq \epsilon\}$. I will prove that

$$\sum_{n=1}^{\infty} \frac{1}{n!z^n}$$

converges uniformly on A_ϵ. Let $z \in A_\epsilon$. Then $|z| \geq \epsilon$ so that

$$\left|\frac{1}{n!z^n}\right| \leq \frac{1}{n!\epsilon^n} = M_n.$$

Since

$$\lim_{n\to\infty} \frac{1/(n+1)!\epsilon^{n+1}}{1/n!\epsilon^n} = \lim_{n\to\infty} \frac{1}{n\epsilon} = 0$$

the ratio test implies that $\sum_{M_n}$ converges. Since $1/n!z^n$ is analytic on A_ϵ for each n, the Weierstrass M-test implies that the series in question converges uniformly and absolutely to an analytic function on A_ϵ. Since every point $z \in \mathbb{C} \setminus \{0\}$ is contained in some A_ϵ, this proves that the series gives us a function that is analytic on all of $z \in \mathbb{C} \setminus \{0\}$.

As to the integral, since the series converges uniformly on $A_{1/2}$ and the unit circle γ is contained in this set, we have

$$\int_\gamma \sum_{n=1}^{\infty} \frac{1}{n!z^n}\, dz = \sum_{n=1}^{\infty} \frac{1}{n!} \int_\gamma \frac{1}{z^n}\, dz = 2\pi i$$

since

$$\int_\gamma \frac{1}{z^n}\, dz$$

is zero unless $n = 1$, in which case we know it's $2\pi i$.

3.1.20 Let $\epsilon > 0$. Since $\{f_n\}$ converge uniformly on the boundary of A, they are uniformly Cauchy there, and so there is an $N \in \mathbb{Z}^+$ so that $|f_n(z) - f_m(z)| < \epsilon$ for all $m, n > N$ and all $z \in \mathrm{bd}(A)$. Since the functions $f_n - f_m$ are analytic on A, and A is bounded, the maximum modulus principle then implies that $|f_n(z) - f_m(z)| < \epsilon$ for all $m, n > N$ and all $z \in A$. Since $\epsilon > 0$ was arbitrary this proves that the sequence $\{f_n\}$ is uniformly Cauchy on A and hence converges uniformly to a function f on A. Since the convergence is uniform on all of A the Analytic Convergence Theorem applies and we conclude that f is analytic on A.

3.2.2

(a) We apply the ratio test:

$$\lim_{n\to\infty}\left|\frac{(n+1)^2 z^{n+1}}{n^2 z^n}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{n^2}|z| = |z|.$$

It follows that the series in question converges (absolutely) when $|z| < 1$ and diverges when $|z| > 1$. Hence, the radius of convergence is 1.

(b) Again, we apply the ratio test:

$$\lim_{n\to\infty}\left|\frac{z^{2(n+1)}/4^{n+1}}{z^{2n}/4^n}\right| = \lim_{n\to\infty}\frac{1}{4}|z|^2 = \frac{|z|^2}{4}.$$

Hence the power series in question will converge (absolutely) for $|z| < 2$ and will diverge for $|z| > 2$. That is, its radius of convergence is 2.

(c) Once more, its the ratio test to the rescue:

$$\lim_{n\to\infty}\left|\frac{(n+1)!z^{n+1}}{n!z^n}\right| = \lim_{n\to\infty} n|z|.$$

This limit is infinite if $z \neq 0$ and is 0 when $z = 0$. Hence, the series converges only when $z = 0$ and so its radius of convergence is 0.

(d) Ratio test, last time:

$$\lim_{n\to\infty}\left|\frac{z^{n+1}/(1+2^{n+1})}{z^n/(1+2^n)}\right| = \lim_{n\to\infty}\frac{1+2^n}{1+2^{n+1}}|z| = \lim_{n\to\infty}\frac{2^{-n}+1}{2^{-n}+2}|z| = \frac{|z|}{2}.$$

As above, this implies the series converges (absolutely) for $|z| < 2$ and diverges for $|z| > 2$, i.e. the radius of convergence is 2.

3.2.4 Let $f(z) = \sin z$. Since $f'(z) = \cos z$, $f''(z) = -\sin z$, $f'''(z) = -\cos z$ and $f^{(4)}(z) = \sin z = f(z)$, it is easy to verify inductively that

$$\begin{aligned}
f^{(4k)}(0) &= \sin 0 = 0 \\
f^{(4k+1)}(0) &= \cos 0 = 1 = (-1)^{2k} \\
f^{(4k+2)}(0) &= -\sin 0 = 0 \\
f^{(4k+3)}(0) &= -\cos 0 = -1 = (-1)^{2k+1}
\end{aligned}$$

for $k \geq 0$. Hence, all of the even power terms in the Taylor series at $z_0 = 0$ vanish while the odd power terms are given by $(-1)^n z^{2n+1}/(2n+1)!$. Since $\sin z$ is entire, its Taylor series at $z_0 = 0$ converges everywhere and we have

$$\sin z = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

for all $z \in \mathbb{C}$.

Because the Taylor series for $\sin z$ has an infinite radius of convergence, it may be differentiated termwise:

$$\cos z = \frac{d}{dz}\sin z = \frac{d}{dz}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}$$

for all $z \in \mathbb{C}$. Uniqueness of power series guarantees that this is the Taylor series for $\cos z$ at $z_0 = 0$.

If $g(z) = (1+z)^\alpha$ $(\alpha \neq 0)$, another easy induction can be used to verify that

$$g^{(n)}(z) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)(1+z)^{\alpha-n}$$

for $n \geq 1$. We note that when α is a positive integer this formula is still valid for $n > \alpha$, since then one of the terms in the product is zero. If we use the principal branch of the logarithm to compute the power function, we obtain

$$g^{(n)}(0) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)$$

for $n \geq 1$ and so the Taylor series for $(1+z)^\alpha$ about $z_0 = 0$ is

$$1 + \sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}z^n = \sum_{n=0}^{\infty}\binom{\alpha}{n}z^n.$$

If α is not a positive integer, then $(1+z)^\alpha$ fails to be analytic at $z = -1$ (because the logarithm does) and so the Taylor series has radius of convergence 1. However, when $\alpha \in \mathbb{Z}^+$ the function $(1+z)^\alpha$ is entire and so the Taylor series (really just a polynomial in this case) has infinite radius of convergence.

3.2.6 It is straightforward to verify that if

$$f(z) = \frac{1}{1+e^z}$$

then:

$$\begin{aligned} f'(z) &= \frac{-e^z}{(1+e^z)^2} \\ f''(z) &= \frac{e^{2z}-e^z}{(1+e^z)^3} \\ f'''(z) &= \frac{-e^{3z}+4e^{2z}-e^z}{(1+e^z)^4}. \end{aligned}$$

Hence

$$\begin{aligned} f(0) &= \frac{1}{2} \\ f'(0) &= \frac{-1}{4} \\ f''(0) &= 0 \\ f'''(0) &= \frac{1}{8} \end{aligned}$$

and so the first four terms of the Taylor expansion of $f(z)$ at $z_0 = 0$ are

$$\frac{1}{2} - \frac{1}{4}z + \frac{1}{48}z^3.$$

The radius of convergence of this Taylor series is the distance from 0 to the nearest point at which $f(z)$ fails to be analytic. Since $z = \pm\pi i$ are equidistant from 0, the radius of convergence is π.

3.2.8

(a) Since $\sin z$ is entire, we can use the Taylor series for $\sin z$ to compute its value at any point. In particular, we have

$$\sin z^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (z^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+2}}{(2n+1)!}.$$

By uniqueness of power series expansions, this must be the Taylor series for $\sin z$ at $z_0 = 0$.

(b) As above, since e^z is entire, we have

$$e^{2z} = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n z^n}{n!}$$

and uniqueness again guarantees this is the Taylor series for e^{2z} at $z_0 = 0$.

3.2.14 Let

$$g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

denote the Taylor series of f around z_0. If $|z - z_0| < R - |z_0|$ then the triangle inequality gives

$$|z| \leq |z - z_0| + |z_0| < R - |z_0| + |z_0| = R.$$

Therefore $z \in A$ (the disk of convergence of f). It follows that $D(z_0, R - |z_0|) \subset A$ and since f is analytic on A, Taylor's theorem implies that g converges to f on $D(z_0, R - |z_0|)$, i.e. the radius of convergence of g is at least $R - |z_0|$. In other words (or other symbols?) $\widetilde{R} \geq R - |z_0|$.

The other inequality is somewhat more difficult to establish. Assume, for the sake of contradiction, that $\widetilde{R} > R + |z_0|$. Let $B = \{z \in \mathbb{C} : |z - z_0| < \widetilde{R}\}$ (i.e. B is the disk of convergence of g) and let $\epsilon = \widetilde{R} - (R + |z_0|) > 0$. I first claim that $D(0, R + \epsilon/2) \subset B$. To prove this, let $z \in D(0, R + \epsilon/2)$. Then

$$|z - z_0| \leq |z| + |z_0| < R + \frac{\epsilon}{2} + |z_0| = R + \frac{\widetilde{R}}{2} - \frac{R}{2} - \frac{|z_0|}{2} + |z_0| = \frac{\widetilde{R}}{2} + \frac{R + |z_0|}{2} < \frac{\widetilde{R}}{2} + \frac{\widetilde{R}}{2} = \widetilde{R}$$

which proves that $z \in B$ and, since $z \in D(0, R + \epsilon/2)$ was arbitrary, establishes the claim.

We will use the claim to arrive at a contradiction. The function g is analytic on B which, as we have just seen, contains $D(0, R + \epsilon/2)$. In particular, if $F(z)$ denotes the Taylor series for $g(z)$ centered at 0, then the radius of convergence of $F(z)$ is at least $R + \epsilon/2$. However, since $A \subset D(0, R + \epsilon/2)$, for $z \in A$ we have $f(z) = g(z)$ and therefore the Taylor series for g at zero is exactly the same as the Taylor series for f at zero. That is, $F(z)$ and $f(z)$ are *the same series.* Since F converges on $D(0, R + \epsilon/2)$, so too then does f. But this contradicts the fact that the radius of convergence of f is exactly R. Hence, or original assumption (that $\widetilde{R} > R + |z_0|$) must be false and we conclude that $\widetilde{R} \leq R + |z_0|$.

3.2.20 Let

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

have radius or convergence $R < \infty$. Let $z, w \in \mathbb{C}$ be on circle of convergence, i.e. let $|w - z_0| = |z - z_0| = R$. Then we have

$$\sum_{n=0}^{\infty} |a_n (w - z_0)^n| = \sum_{n=0}^{\infty} |a_n||w - z_0|^n = \sum_{n=0}^{\infty} |a_n| R^n = \sum_{n=0}^{\infty} |a_n||z - z_0|^n = \sum_{n=0}^{\infty} |a_n (z - z_0)^n|.$$

Therefore, if the series converges absolutely at z it converges absolutely at w as well. It follows that if the power series converges absolutely at a single point on the circle of convergence then it converges absolutely at all such points. Hence, the series converges absolutely nowhere or everywhere on the circle of convergence.

As examples of each situation, consider the two power series

$$\sum_{n=1}^{\infty} \frac{z^n}{n}$$

and

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

Both series have radius of convergence 1, as is readily verified using the ratio test, but the former does not converge absolutely when $|z| = 1$ (it becomes the harmonic series in that case) and the latter does converge absolutely when $|z| = 1$ (it becomes a p-series in that case with $p = 2$).

3.2.24 Since the series for $\zeta(z)$ converges uniformly on closed disks in the half plane $\{\operatorname{Re} z > 1\}$, we can differentiate the series term-wise to compute the derivatives necessary for the Taylor expansion. Indeed, we have:

$$\begin{aligned}
\frac{d^k}{dz^k}\zeta(z) &= \frac{d^k}{dz^k}\sum_{n=1}^{\infty} n^{-z} \\
&= \sum_{n=1}^{\infty} \frac{d^k}{dz^k} n^{-z} \\
&= \sum_{n=1}^{\infty} \frac{d^k}{dz^k} e^{-z\log n} \\
&= \sum_{n=1}^{\infty} (-\log n)^k \, e^{-z\log n} \\
&= \sum_{n=1}^{\infty} (-\log n)^k \, n^{-z}
\end{aligned}$$

for $\operatorname{Re} z > 1$. Therefore

$$\zeta^{(k)}(2) = \sum_{n=1}^{\infty} (-\log n)^k \, n^{-2}$$

and the Taylor series is

$$\sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{n=1}^{\infty} (-\log n)^k \, n^{-2} \right) (z-2)^k.$$