Exercise 1. Interpret, and then verify, the following equalities. Throughout, $g(x)$ and $h(x)$ denote positive-valued functions.

a. $O(g(x)) + O(h(x)) = O(\max\{g(x), h(x)\})$.

b. $g(x)O(h(x)) = O(g(x)h(x)) = O(g(x))\,O(h(x))$.

c. $O(O(g(x))) = O(g(x))$.

Exercise 2.

a. Let $f(x)$ and $g(x)$ be polynomials with real coefficients of degrees m and n, respectively. Prove that
$$\lim_{x\to\infty} \frac{f(x)x^{n-m}}{g(x)}$$
exists and is nonzero.

b. Prove that
$$\frac{f(x)}{g(x)} = O\left(x^{m-n}\right)$$
for all sufficiently large x. What does the implied constant depend on?

Exercise 3.

a. Let $k, \epsilon > 0$. Prove that
$$(\log x)^k = O(x^\epsilon)$$
for all sufficiently large x. Does the implied constant depend on k or ϵ?

b. Let $f(x)$ be a polynomial with real coefficients. Prove that for any $\epsilon > 0$
$$f(\log x) = O(x^\epsilon)$$
for all sufficiently large x. What does the implied constant depend on?

Exercise 4. Suppose that $f(x)$ and $g(x)$ are continuous for $x \geq 1$, that $g(x) > 0$, and that $f(x) = O(g(x))$ for all sufficiently large x. Prove that $f(x) = O(g(x))$ for *all* $x \geq 1$, but that the implied constant will depend on $f(x)$ and $g(x)$. [*Hint:* A continuous function on a closed interval attains its maximum and minimum values.]