Math 4391 Fall 2008

Number Theory II

Final Exam

Due Tuesday, December 16, 5:00 pm

Your name (please print):

Instructions: This is an untimed take home exam. You may freely consult your lecture notes, homework and course textbook (indeed, you are fully expected to), but no other resources are permitted. You must justify all of your answers to receive credit, and you must carefully cite any results that you choose to quote (e.g. "By Theorem 2.18..." or "We proved in class that..."). Be sure to staple this page to the front of your exam solutions.

The Honor Code requires that you neither give nor receive any aid on this exam.

If you are bound by the Academic Honor Code, please indicate that you have read and understood these guidelines by signing your name in the space provided:

Pledged: ______________________________

Do not write below this line

Problem	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Points	5	10	10	10	15	5	10	10	10	15
Score										

Total:____________________

1. Let $\{a_n\}$ be a sequence of real numbers satisfying

$$A(x) = \sum_{n \le x} a_n = O(x^c).$$

Prove that the series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges for $s > c$.

2. Consider the following statement: for all $h, k \in \mathbb{N}$ with $(h, k) = 1$ there exists a prime number p so that $p \equiv h \pmod{k}$. Prove that this statement implies Dirichlet's theorem.

3. Let χ be a Dirichlet character mod k.

a. If k is squarefree, prove that k is the smallest positive period of χ.

b. If $k = 2p$, p an odd prime, and χ is nonprincipal, prove that χ has conductor p.

4. Let χ be the Dirichlet character mod 10 satisfying $\chi(3) = -i$. Compute $L(1, \chi)$. Express your answer in terms of radicals and i only. You may find it useful to know that

$$\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}, \quad \cos\left(\frac{3\pi}{5}\right) = \frac{1-\sqrt{5}}{4}.$$

5. Let χ be a nonprincipal Dirichlet character mod k satisfying $\chi(-1) = 1$.

a. Prove that

$$\sum_{m=1}^{\left[\frac{k+1}{2}\right]-1} \chi(m) = 0.$$

[*Suggestion:* Begin with the sum up to k and then pair appropriate terms. It might be helpful to consider the k even and k odd cases separately.]

b. Prove that

$$\sum_{m=1}^{k} \chi(m)m = 0.$$

6. Show that

$$\lim_{x \to \infty} \sum_{\sqrt{x} < p \le x} \frac{1}{p} = \log 2.$$

7. If $s > 0$, $s \neq 1$, prove that

$$\sum_{n \le x} \frac{d(n)}{n^s} = \frac{x^{1-s} \log x}{1-s} + \zeta(s)^2 + O\left(x^{1-s}\right).$$

Use this to prove that the sum

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^2}$$

converges and find its value.

8. Prove that there is a constant B so that for $x \geq 2$

$$\sum_{2 \leq n \leq x} \frac{1}{n \log n} = \log\log x + B + O\left(\frac{1}{x \log x}\right).$$

9. Prove that the sum $\displaystyle\sum_{p} \frac{1}{p \log p}$ converges.

10. Fix $k \in \mathbb{N}$ with $k \geq 2$. Call an integer $n \in \mathbb{N}$ *k-power free* if it is not divisible by the kth power of any prime.

a. Show that every $n \in \mathbb{N}$ can be written uniquely as $n = a^k b$ where b is k-power free.

b. Given $n \in \mathbb{N}$ write $n = a^k b$ as in part (a) and set $F_k(n) = b$. Prove that

$$\sum_{d|n} \mu\left(F_k(d)\right) = \begin{cases} 1 & \text{if } n \text{ is a } k\text{th power,} \\ 0 & \text{otherwise.} \end{cases}$$

c. Deduce that

$$\mu\left(F_k(n)\right) = \sum_{d^k|n} \mu\left(\frac{n}{d^k}\right)$$

and

$$\sum_{n \leq x} \mu\left(F_k(n)\right)\left[\frac{x}{n}\right] = \left[\sqrt[k]{x}\right].$$