Number Theory I | Assignment 3.2
Spring 2012 | Due January 31

Exercise 1. Let $p \geq 2$. Suppose that p has the following property: for any a and b, if $p|ab$ then $p|a$ or $p|b$. Prove that p is prime. [*Suggestion:* Argue by contradiction.]

Exercise 2. Let p be a prime and let n be nonzero. Prove that there exist a unique $k \geq 0$ and nonzero m so that $n = p^k m$ and $p \nmid m$. The number k is called the *p-adic valuation of* n and is denoted $\nu_p(n)$.

Exercise 3. Prove that the p-adic valuation has the following properties. For all nonzero m and n:

- **a.** $\nu_p(mn) = \nu_p(m) + \nu_p(n)$.
- **b.** $\nu_p(m+n) \geq \min\{\nu_p(m), \nu_p(n)\}$ (provided $m + n \neq 0$).
- **c.** The inequality in **b.** is actually an equality in the case that $\nu_p(m) \neq \nu_p(n)$.

[*Remark:* The uniqueness statement in Exercise 1 is useful here, for it tells us that any time we write $n = p^k m$ with $k \geq 0$ and $p \nmid m$, we *must* have $k = \nu_p(n)$.]

Exercise 4. Let $p_1, p_2, \ldots, p_r$ be distinct primes and let $a_1, a_2, \ldots, a_r$ be nonnegative. Let $k \geq 2$. If

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r},$$

prove that n is a k^{th} power (i.e. $n = m^k$ for some integer m) if and only if $k|a_i$ for all i.

Exercise 5. An integer n is called *square-free* if it is not divisible by the square of any integer greater that 1.

- **a.** Prove that n is square-free if and only if n can be factored into a product of distinct primes.
- **b.** Prove that any integer $n > 1$ is the product of a square-free integer and a perfect square.