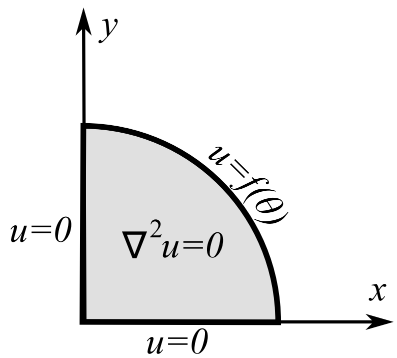
Partial Differential Equations Final Exam Review Exercises
Spring 2012

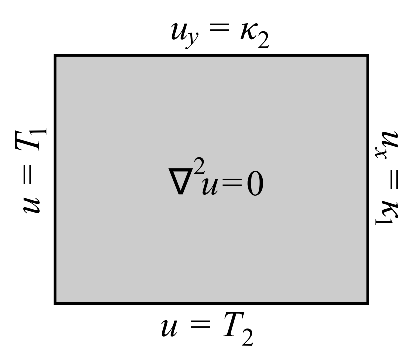
These problems are primarily exercises in the method of separation of variables and Sturm-Liouville theory, which are arguably the most important topics that were presented this semester. Although we did briefly cover other topics, you are responsible for reviewing and practicing those on your own.

Exercise 1. Solve the following Dirichlet problem on a quarter-disk of radius a:



[*Suggestions:* 1. Use polar coordinates. 2. An ODE of the form $x^2y'' + axy' + by = 0$ is called an *Euler equation*. Its solution is treated in the appendix to our textbook.]

Exercise 2. Find the steady state heat distribution in an $a \times b$ plate, if we keep temperatures of the left and bottom edges constant, while we hold the heat flux at the other two edges constant (see below).



You may assume that κ_1 and κ_2 are both positive.

Exercise 3. [5% Extra Credit] Consider the preceding problem.

a. Write a Maple file that accepts as (easily modified) inputs the values of a, b, T_1, T_2, κ_1 and κ_2 and produces a graph of the steady state temperature in the plate.

b. The steady state temperature can be approximated numerically (on a specified grid of points) using the *finite difference method*, a technique which you will study in a later course. The Maple file **laplace_data.mw** includes an implementation of this method, with specific choices of the parameters a, b, T_1, T_2, κ_1 and κ_2. Use the code you developed in part **a** to compare the values of the analytic steady state solution to those provided by the finite difference solution.

Exercise 4. A rod of length L is given an initial temperature profile $f(x)$ along its length, is held at a constant temperature T at its right end, and radiates heat at a rate proportional to its temperature at the other end. If $u(x,t)$ denotes the temperature of the rod at position x and time t, this situation can be modeled by the boundary value problem

$$\begin{aligned}
&u_t = c^2 u_{xx}, \quad t > 0, \quad 0 < x < L, \\
&u_x(0,t) = \kappa u(0,t), \quad u(L,t) = 100, \quad t > 0, \\
&u(x,0) = f(x), \quad 0 \le x \le L,
\end{aligned}$$

where $\kappa > 0$.

a. Find the steady state temperature distribution in the bar.

b. Subtract the steady state from the original problem, and solve the resulting problem.

c. Give the general solution to the original problem.

Exercise 5. In the presence of resistance proportional to velocity (damping), the one dimensional wave equation becomes

$$u_{tt} + 2k u_t = c^2 u_{xx},$$

where $k > 0$. Assume we have a damped vibrating string of length L with fixed endpoints.

a. Separate variables and find the normal modes of the damped vibrating string.

b. Verify that the normal modes all tend to zero as $t \to \infty$.