The one dimensional heat equation: Neumann and Robin boundary conditions

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Partial Differential Equations
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The heat equation with Neumann boundary conditions

Our goal is to solve:

$$u_t = c^2 u_{xx}, \qquad 0 < x < L \,,\, 0 < t, \qquad (1)$$

$$u_x(0,t) = u_x(L,t) = 0, \qquad 0 < t, \qquad (2)$$

$$u(x,0) = f(x), \qquad 0 < x < L. \qquad (3)$$

As before, we will use **separation of variables** to find a family of simple solutions to (1) and (2), and then the **principle of superposition** to construct a solution satisfying (3).

Separation of variables

Assuming that $u(x,t) = X(x)T(t)$, the heat equation (1) becomes

$$XT' = c^2X''T.$$

This implies

$$\frac{X''}{X} = \frac{T'}{c^2T} = k,$$

which we write as

$$X'' - kX = 0, \qquad (4)$$

$$T' - c^2kT = 0. \qquad (5)$$

The initial conditions (2) become $X'(0)T(t) = X'(L)T(t) = 0$, or

$$X'(0) = X'(L) = 0. \qquad (6)$$

Case 1: $k = \mu^2 > 0$

The ODE (4) becomes $X'' - \mu^2 X = 0$ with general solution

$$X = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$

The boundary conditions (6) are

$$\begin{aligned} 0 &= X'(0) = \mu c_1 - \mu c_2 = \mu(c_1 - c_2), \\ 0 &= X'(L) = \mu c_1 e^{\mu L} - \mu c_2 e^{-\mu L} = \mu(c_1 e^{\mu L} - c_2 e^{-\mu L}). \end{aligned}$$

The first gives $c_1 = c_2$. When we substitute this into the second we get

$$2 c_1 \mu \sinh \mu L = 0.$$

Since $\mu, L > 0$, we must have $c_1 = c_2 = 0$. Hence $X = 0$, i.e. *there are only trivial solutions in the case $k > 0$.*

Case 2: $k = 0$

The ODE (4) is simply $X'' = 0$ so that

$$X = Ax + B.$$

The boundary conditions (6) yield

$$0 = X'(0) = X'(L) = A.$$

Taking $B = 1$ we get the solution

$$X_0 = 1.$$

The corresponding equation (5) for T is $T' = 0$, which yields $T = C$. We set

$$T_0 = 1.$$

These give the zeroth **normal mode**:

$$\boxed{u_0(x,t) = X_0(x)T_0(t) = 1.}$$

Case 3: $k = -\mu^2 < 0$

The ODE (4) is now $X'' + \mu^2 X = 0$ with solutions

$$X = c_1 \cos \mu x + c_2 \sin \mu x.$$

The boundary conditions (6) yield

$$\begin{aligned} 0 &= X'(0) = -\mu c_1 \sin 0 + \mu c_2 \cos 0 = \mu c_2, \\ 0 &= X'(L) = -\mu c_1 \sin \mu L + \mu c_2 \cos \mu L. \end{aligned}$$

The first of these gives $c_2 = 0$. In order for X to be nontrivial, the second shows that we also need

$$\sin \mu L = 0.$$

Case 3: $k = -\mu^2 < 0$

This can occur if and only if $\mu L = n\pi$, that is

$$\mu = \mu_n = \frac{n\pi}{L}, \ \ n = \pm 1, \pm 2, \pm 3, \ldots$$

Choosing $c_1 = 1$ yields the solutions

$$X_n = \cos \mu_n x, \ \ n = 1, 2, 3, \ldots$$

For each n the corresponding equation (5) for T becomes $T' = -\lambda_n^2 T$, with $\lambda_n = c\mu_n$. Up to a constant multiple, the solution is

$$T_n = e^{-\lambda_n^2 t}.$$

Normal modes and superposition

Multiplying these together gives the n**th normal mode**

$$\boxed{u_n(x,t) = X_n(x)T_n(t) = e^{-\lambda_n^2 t}\cos\mu_n x, \;\; n = 1,2,3,\ldots}$$

where $\mu_n = n\pi/L$ and $\lambda_n = c\mu_n$.

The principle of superposition now guarantees that for any choice of constants $a_0, a_1, a_2, \ldots$

$$u(x,t) = a_0 u_0 + \sum_{n=1}^{\infty} a_n u_n = a_0 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t}\cos\mu_n x \qquad (7)$$

is a solution of the heat equation (1) with the Neumann boundary conditions (2).

Initial conditions

If we now require that the solution (7) satisfy the initial condition (3) we find that we need

$$f(x) = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L},\ \ 0 < x < L.$$

This is simply the *cosine series expansion of* $f(x)$. Using our previous results, we finally find that if $f(x)$ is piecewise smooth then

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx,\ \ a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx,\ \ n \geq 1.$$

Conclusion

Theorem

If $f(x)$ is piecewise smooth, the solution to the heat equation (1) with Neumann boundary conditions (2) and initial conditions (3) is given by

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t} \cos \mu_n x,$$

where

$$\mu_n = \frac{n\pi}{L}, \ \lambda_n = c\mu_n,$$

and the coefficients $a_0, a_1, a_2, \ldots$ are those occurring in the cosine series expansion of $f(x)$. They are given explicitly by

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \ a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \ n \geq 1.$$

Example 1

Example

Solve the following heat conduction problem:

$$\begin{aligned} u_t &= \frac{1}{4} u_{xx}, && 0 < x < 1 \ , \ 0 < t, \\ u_x(0,t) &= u_x(1,t) = 0, && 0 < t, \\ u(x,0) &= 100x(1-x), && 0 < x < 1. \end{aligned}$$

With $L = 1$, $\mu_n = n\pi$ and $f(x) = 100x(1-x)$ we find

$$a_0 = \int_0^1 100x(1-x)\,dx = \frac{50}{3}$$

$$a_n = 2\int_0^1 100x(1-x)\cos n\pi x\,dx = \frac{-200(1+(-1)^n)}{n^2\pi^2}, \;\; n \geq 1.$$

Example 1

Since $c = 1/2$, $\lambda_n = n\pi/2$. Plugging everything into our general solution we get

$$\boxed{u(x,t) = \frac{50}{3} - \frac{200}{\pi^2}\sum_{n=1}^{\infty}\frac{(1+(-1)^n)}{n^2}e^{-n^2\pi^2 t/4}\cos n\pi x.}$$

As in the case of Dirichlet boundary conditions, the exponential terms decay rapidly with t. We have

$$\lim_{t\to\infty} u(x,t) = \frac{50}{3}.$$

Remarks

- At any given time, the *average temperature* in the bar is

$$\overline{u}(t) = \frac{1}{L}\int_0^L u(x,t)\,dx.$$

- In the case of Neumann boundary conditions, one has

$$\overline{u}(t) = a_0 = \overline{f}.$$

  That is, *the average temperature is constant* and *is equal to the initial average temperature.*

- Also in this case

$$\lim_{t\to\infty} u(x,t) = a_0$$

  for all x. That is, at any point in the bar *the temperature tends to the initial average temperature.*

The heat equation with Robin boundary conditions

We now consider the problem

$$
\begin{aligned}
u_t &= c^2 u_{xx}, && 0 < x < L\ ,\ 0 < t, \\
u(0,t) &= 0, && 0 < t, && (8)\\
u_x(L,t) &= -\kappa u(L,t), && 0 < t, && (9)\\
u(x,0) &= f(x), && 0 < x < L.
\end{aligned}
$$

- In (9) we take $\kappa > 0$. This states that the bar radiates heat to its surroundings at a rate proportional to its current temperature.
- Recall that conditions such as (9) are called **Robin conditions**.

Separation of variables

As before, the assumption that $u(x,t) = X(x)T(t)$ leads to the ODEs

$$X'' - kX = 0,$$
$$T' - c^2 kT = 0,$$

and the boundary conditions (8) and (9) imply

$$X(0) = 0,$$
$$X'(L) = -\kappa X(L).$$

Also as before, the possibilities for X depend on the sign of the separation constant k.

Case 1: $k = 0$

We have $X'' = 0$ and so $X = Ax + B$ with

$$\begin{aligned} 0 &= X(0) = B, \\ A &= X'(L) = -\kappa X(L) = -\kappa(AL + B). \end{aligned}$$

Together these give $A(1 + \kappa L) = 0$. Since $\kappa, L > 0$, we have $A = 0$ and hence $X = 0$. Thus,

there are only trivial solutions in this case.

Case 2: $k = \mu^2 > 0$

Once again we have $X'' - \mu^2 X = 0$ and

$$X = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$

The boundary conditions become

$$\begin{aligned} 0 &= c_1 + c_2, \\ \mu(c_1 e^{\mu L} - c_2 e^{-\mu L}) &= -\kappa(c_1 e^{\mu L} + c_2 e^{-\mu L}). \end{aligned}$$

The first gives $c_2 = -c_1$, which when substituted in the second yields

$$2\mu c_1 \cosh \mu L = -2\kappa c_1 \sinh \mu L.$$

Case 2: $k = \mu^2 > 0$

We may rewrite this as

$$c_1 \left(\mu \cosh \mu L + \kappa \sinh \mu L\right) = 0.$$

The quantity in parentheses is positive (since μ, κ and L are), so this means we must have $c_1 = -c_2 = 0$. Hence $X = 0$ and

there are only trivial solutions in this case.

Case 3: $k = -\mu^2 < 0$

From $X'' + \mu^2 X = 0$ we find

$$X = c_1 \cos \mu x + c_2 \sin \mu x$$

and from the boundary conditions we have

$$\begin{aligned} 0 &= c_1, \\ \mu(-c_1 \sin \mu L + c_2 \cos \mu L) &= -\kappa(c_1 \cos \mu L + c_2 \sin \mu L). \end{aligned}$$

Together these imply that

$$c_2 \left(\mu \cos \mu L + \kappa \sin \mu L\right) = 0.$$

Case 3: $k = -\mu^2 < 0$

Since we want nontrivial solutions (i.e. $c_2 \neq 0$), we must have

$$\mu \cos \mu L + \kappa \sin \mu L = 0$$

which can be rewritten as

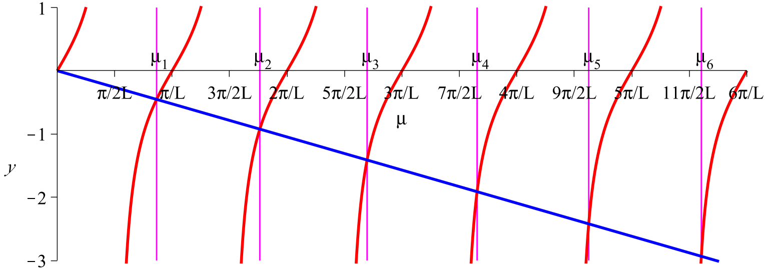
$$\tan \mu L = -\frac{\mu}{\kappa}.$$

This equation has an infinite sequence of positive solutions

$$0 < \mu_1 < \mu_2 < \mu_3 < \cdots$$

Case 3: $k = -\mu^2 < 0$

The figure below shows the curves $y = \tan \mu L$ (in red) and $y = -\mu/\kappa$ (in blue).



The μ-coordinates of their intersections (in pink) are the values μ_1, μ_2, μ_3, ...

Remarks

From the diagram we see that:

- For each n

$$(2n-1)\pi/2L < \mu_n < n\pi/L.$$

- As $n \to \infty$

$$\mu_n \to (2n-1)\pi/2L.$$

- Smaller values of κ and L tend to accelerate this convergence.

Normal modes

As in the earlier situations, for each $n \geq 1$ we have the solution

$$X_n = \sin \mu_n x$$

and the corresponding

$$T_n = e^{-\lambda_n^2 t}, \ \lambda_n = c\mu_n$$

which give the **normal modes** of the heat equation with boundary conditions (8) and (9)

$$\boxed{u_n(x,t) = X_n(x) T_n(t) = e^{-\lambda_n^2 t} \sin \mu_n x.}$$

Superposition

Superposition of normal modes gives the **general solution**

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n^2 t} \sin \mu_n x.$$

Imposing the initial condition $u(x,0) = f(x)$ gives us

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \mu_n x.$$

This is a **generalized Fourier series** for $f(x)$. It is *different* from the ordinary sine series for $f(x)$ since

μ_n is not a multiple of a common value.

Generalized Fourier coefficients

To compute the **generalized Fourier coefficients** c_n we will use the following fact.

Proposition

The functions

$$X_1(x) = \sin \mu_1 x, X_2(x) = \sin \mu_2 x, X_3(x) = \sin \mu_3 x, \ldots$$

form a complete orthogonal set on $[0, L]$.

- *Complete* means that all "sufficiently nice" functions can be represented via generalized Fourier series. This is a consequence of **Sturm-Liouville theory**, which we will study later.
- We can verify orthogonality directly, and will use this to express the coefficients c_n as ratios of inner products.

Generalized Fourier coefficients

Assuming orthogonality for the moment, for any $n \geq 1$ we have the familiar computation

$$\begin{aligned}
\langle f, X_n \rangle &= \left\langle \sum_{m=1}^{\infty} c_m \sin \mu_m x, \sin \mu_n x \right\rangle \\
&= \sum_{m=1}^{\infty} c_m \langle \sin \mu_m x, \sin \mu_n x \rangle \\
&= c_n \langle \sin \mu_n x, \sin \mu_n x \rangle \\
&= c_n \langle X_n, X_n \rangle
\end{aligned}$$

since the inner products with $m \neq n$ all equal zero.

Generalized Fourier coefficients

It follows immediately that the generalized Fourier coefficients are given by

$$c_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle} = \frac{\displaystyle\int_0^L f(x) \sin \mu_n x \, dx}{\displaystyle\int_0^L \sin^2 \mu_n x \, dx}.$$

- For any given $f(x)$ these integrals can typically be computed explicitly in terms of μ_n.
- The values of μ_n, however, must typically be found via numerical methods.

Conclusion

Theorem

The solution to the heat equation (1) with Robin boundary conditions (8) and (9) and initial condition (3) is given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n^2 t} \sin \mu_n x,$$

where μ_n is the nth positive solution to

$$\tan \mu L = \frac{-\mu}{\kappa},$$

$\lambda_n = c\mu_n$*, and the coefficients c_n are given by*

$$c_n = \frac{\int_0^L f(x) \sin \mu_n x \, dx}{\int_0^L \sin^2 \mu_n x \, dx}.$$

Example 2

Example

Solve the following heat conduction problem:

$$u_t = \frac{1}{25} u_{xx}, \qquad 0 < x < 3 \ , \ 0 < t,$$
$$u(0,t) = 0, \qquad 0 < t,$$
$$u_x(3,t) = -\frac{1}{2} u(3,t), \qquad 0 < t,$$
$$u(x,0) = 100\left(1 - \frac{x}{3}\right), \qquad 0 < x < 3.$$

We have $c = 1/5$, $L = 3$, $\kappa = 1/2$ and $f(x) = 100(1 - x/3)$.

Example 2

The integrals defining the Fourier coefficients are

$$100\int_0^3 \left(1-\frac{x}{3}\right)\sin \mu_n x\,dx = \frac{100(3\mu_n - \sin 3\mu_n)}{3\mu_n^2}$$

and

$$\int_0^3 \sin^2 \mu_n x\,dx = \frac{3}{2} + \cos^2 3\mu_n.$$

Hence

$$c_n = \frac{200(3\mu_n - \sin 3\mu_n)}{3\mu_n^2\,(3 + 2\cos^2 3\mu_n)}.$$

Example 2

We can therefore write out the full solution as

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{200(3\mu_n - \sin 3\mu_n)}{3\mu_n^2\left(3 + 2\cos^2 3\mu_n\right)} e^{-\mu_n^2 t/25} \sin \mu_n x,}$$

where μ_n is the nth positive solution to $\tan 3\mu = -2\mu$.

Example 2

Remarks:

- In order to use this solution for numerical approximation or visualization, we must compute the values μ_n.
- This can be done numerically in Maple, using the `fsolve` command. Specifically, μ_n can be computed via the command

```
fsolve(tan(m*L)=-m/k,m=(2*n-1)*Pi/(2*L)..n*Pi/L);
```

  where `L` and `k` have been assigned the values of L and κ, respectively.
- These values can be computed and stored in an `Array` structure, or one can define μ_n as a function using the `->` operator.

Example 2

Here are approximate values for the first 5 values of μ_n and c_n.

n	μ_n	c_n
1	0.7249	47.0449
2	1.6679	45.1413
3	2.6795	21.3586
4	3.7098	19.3403
5	4.7474	12.9674

Therefore

$$u(x,t) = 47.0449e^{-0.0210t}\sin(0.7249x) + 45.1413e^{-0.1113t}\sin(1.6679x)$$
$$+ 21.3586e^{-0.2872t}\sin(2.6795x) + 19.3403e^{-0.5505t}\sin(3.7098x)$$
$$+ 12.9674e^{-0.9015t}\sin(4.7474x) + \cdots$$