The two dimensional wave equation

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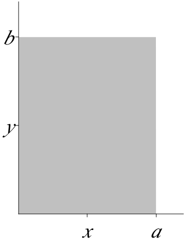
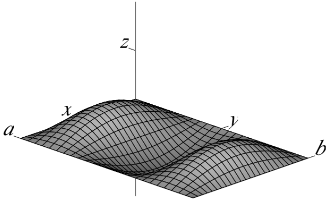
Physical motivation

Consider a thin elastic membrane stretched tightly over a rectangular frame. Suppose the dimensions of the frame are $a \times b$ and that we keep the edges of the membrane fixed to the frame.





- Perturbing the membrane from equilibrium results in some sort of vibration of the surface.
- Our goal is to mathematically model the vibrations of the membrane surface.

We let

$$u(x,y,t) = \begin{array}{l}\text{deflection of membrane from equilibrium at} \\ \text{position } (x,y) \text{ and time } t.\end{array}$$

For a fixed t, the surface $z = u(x,y,t)$ gives the shape of the membrane at time t.

Under ideal assumptions (e.g. uniform membrane density, uniform tension, no resistance to motion, small deflection, etc.) one can show that u satisfies the **two dimensional wave equation**

$$\boxed{u_{tt} = c^2\nabla^2 u = c^2(u_{xx} + u_{yy})} \tag{1}$$

for $0 < x < a$, $0 < y < b$.

Remarks:

- For the derivation of the wave equation from Newton's second law, see exercise 3.2.8.
- As in the one dimensional situation, the constant c has the units of velocity. It is given by

$$c^2 = \frac{\tau}{\rho},$$

  where τ is the tension per unit length, and ρ is mass density.
- The operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

  is called the **Laplacian**. It will appear in many of our subsequent investigations.

The fact that we are keeping the edges of the membrane fixed is expressed by the **boundary conditions**

$$\begin{aligned} u(0,y,t) = u(a,y,t) = 0, &\qquad 0 \le y \le b,\ t \ge 0, \\ u(x,0,t) = u(x,b,t) = 0, &\qquad 0 \le x \le a,\ t \ge 0. \end{aligned} \tag{2}$$

We must also specify how the membrane is initially deformed and set into motion. This is done via the **initial conditions**

$$\begin{aligned} u(x,y,0) = f(x,y), &\qquad (x,y) \in R, \\ u_t(x,y,0) = g(x,y), &\qquad (x,y) \in R, \end{aligned} \tag{3}$$

where $R = [0,a] \times [0,b]$.

Solving the 2D wave equation

Goal: Write down a solution to the wave equation (1) subject to the boundary conditions (2) and initial conditions (3).

We will follow the (hopefully!) familiar process of

- using **separation of variables** to produce simple solutions to (1) and (2),
- and then the **principle of superposition** to build up a solution that satisfies (3) as well.

Separation of variables

We seek nontrivial solutions of the form

$$u(x,y,t) = X(x)Y(y)T(t).$$

Plugging this into the wave equation (1) we get

$$XYT'' = c^2\left(X''YT + XY''T\right).$$

If we divide both sides by c^2XYT this becomes

$$\frac{T''}{c^2T} = \frac{X''}{X} + \frac{Y''}{Y}.$$

Because the two sides are functions of different independent variables, they must be constant:

$$\frac{T''}{c^2T} = A = \frac{X''}{X} + \frac{Y''}{Y}.$$

The first equality becomes

$$T'' - c^2AT = 0.$$

The second can be rewritten as

$$\frac{X''}{X} = -\frac{Y''}{Y} + A.$$

Once again, the two sides involve unrelated variables, so both are constant:

$$\frac{X''}{X} = B = -\frac{Y''}{Y} + A.$$

If we let $C = A - B$ these equations can be rewritten as

$$X'' - BX = 0,$$
$$Y'' - CY = 0.$$

The first boundary condition is

$$0 = u(0, y, t) = X(0)Y(y)T(t),\ 0 \leq y \leq b,\ t \geq 0.$$

Since we want nontrivial solutions only, we can cancel Y and T, yielding

$$X(0) = 0.$$

When we perform similar computations with the other three boundary conditions we also get

$$\begin{aligned} X(a) &= 0, \\ Y(0) = Y(b) &= 0. \end{aligned}$$

There are no boundary conditions on T.

Fortunately, we have already solved the two boundary value problems for X and Y. The nontrivial solutions are

$$\begin{aligned} X_m(x) &= \sin \mu_m x, & \mu_m &= \frac{m\pi}{a}, & m &= 1, 2, 3, \ldots \\ Y_n(y) &= \sin \nu_n y, & \nu_n &= \frac{n\pi}{b}, & n &= 1, 2, 3, \ldots \end{aligned}$$

with separation constants $B = -\mu_m^2$ and $C = -\nu_n^2$.

Recall that T must satisfy

$$T'' - c^2 AT = 0$$

with $A = B + C = -\left(\mu_m^2 + \nu_n^2\right) < 0$. It follows that for any choice of m and n the general solution for T is

$$T_{mn}(t) = B_{mn} \cos \lambda_{mn} t + B_{mn}^* \sin \lambda_{mn} t,$$

where

$$\lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

These are the **characteristic frequencies** of the membrane.

Assembling our results, we find that for any *pair* $m, n \geq 1$ we have the **normal mode**

$$\boxed{\begin{aligned} u_{mn}(x,y,t) &= X_m(x)Y_n(y)T_{mn}(t) \\ &= \sin\mu_m x \sin\nu_n y \left(B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t\right) \end{aligned}}$$

where

$$\boxed{\mu_m = \frac{m\pi}{a},\ \nu_n = \frac{n\pi}{b},\ \lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2}.}$$

Remarks:

- Note that the normal modes:
  - oscillate spatially with frequency μ_m in the x-direction,
  - oscillate spatially with frequency ν_n in the y-direction,
  - oscillate in time with frequency λ_{mn}.
- While μ_m and ν_n are simply multiples of π/a and π/b, respectively,

  λ_{mn} *is not a multiple of any basic frequency.*

Superposition

According to the principle of superposition, because the normal modes u_{mn} satisfy the homogeneous conditions (1) and (2), we may add them to obtain the general solution

$$\boxed{u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\mu_m x\sin\nu_n y\left(B_{mn}\cos\lambda_{mn}t+B_{mn}^*\sin\lambda_{mn}t\right).}$$

We must use a double series since the indices m and n vary independently throughout the set

$$\mathbb{N}=\{1,2,3,\ldots\}.$$

Initial conditions

Finally, we must determine the values of the coefficients B_{mn} and B^*_{mn} that are required so that our solution also satisfies the initial conditions (3). The first of these is

$$f(x,y) = u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{mn} \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y$$

and the second is

$$g(x,y) = u_t(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \lambda_{mn} B^*_{mn} \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y.$$

These are examples of **double Fourier series**.

Orthogonality (again!)

To compute the coefficients in a double Fourier series we can appeal to the following result.

Theorem

The functions

$$Z_{mn}(x,y) = sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y, \; m,n \in \mathbb{N}$$

are pairwise orthogonal relative to the inner product

$$\langle f,g\rangle = \int_0^a \int_0^b f(x,y)g(x,y)\,dy\,dx.$$

This is easily verified using the orthogonality of the functions $\sin(n\pi x/a)$ on the interval $[0,a]$.

Using the usual argument, it follows that *assuming* we can write

$$f(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y,$$

then

$$B_{mn}=\frac{\langle f,Z_{mn}\rangle}{\langle Z_{mn},Z_{mn}\rangle}=\frac{\displaystyle\int_0^a\int_0^b f(x,y)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,dy\,dx}{\displaystyle\int_0^a\int_0^b \sin^2\frac{m\pi}{a}x\sin^2\frac{n\pi}{b}y\,dy\,dx}$$

$$=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,dy\,dx \tag{4}$$

Representability

The question of whether or not a given function is equal to a double Fourier series is partially answered by the following result.

Theorem

If $f(x, y)$ is a C^2 function on the rectangle $[0, a] \times [0, b]$, then

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y,$$

where B_{mn} is given by (4).

- To say that $f(x, y)$ is a C^2 *function* means that f as well as its first and second order partial derivatives are all continuous.
- While not as general as the Fourier representation theorem, this result is sufficient for our applications.

Conclusion

Theorem

Suppose that $f(x,y)$ and $g(x,y)$ are C^2 functions on the rectangle $[0,a]\times[0,b]$. The solution to the wave equation (1) with boundary conditions (2) and initial conditions (3) is given by

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\mu_m x\sin\nu_n y\left(B_{mn}\cos\lambda_{mn}t+B_{mn}^{*}\sin\lambda_{mn}t\right)$$

where

$$\mu_m=\frac{m\pi}{a},\ \nu_n=\frac{n\pi}{b},\ \lambda_{mn}=c\sqrt{\mu_m^2+\nu_n^2},$$

Theorem (continued)

and the coefficients B_{mn} *and* B^*_{mn} *are given by*

$$B_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,dy\,dx$$

and

$$B^*_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^a\int_0^b g(x,y)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,dy\,dx.$$

- We have not actually verified that this solution is *unique*, i.e. that this is the *only* solution to the wave equation with the given boundary and initial conditions.
- Uniqueness can be proven using an argument involving **conservation of energy** in the vibrating membrane.

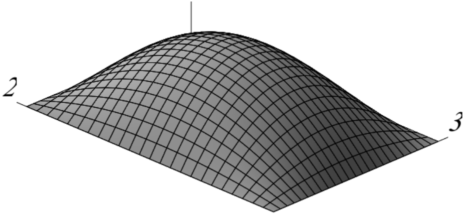
Example 1

Example

A 2×3 *rectangular membrane has* $c = 6$*. If we deform it to have shape given by*

$$f(x, y) = xy(2 - x)(3 - y),$$

keep its edges fixed, and release it at $t = 0$*, find an expression that gives the shape of the membrane for* $t > 0$*.*



We must compute the coefficients B_{mn} and B^*_{mn}. Since $g(x,y)=0$ we immediately have

$$B^*_{mn}=0.$$

We also have

$$\begin{aligned}
B_{mn} &= \frac{4}{2\cdot 3}\int_0^2\int_0^3 xy(2-x)(3-y)\sin\frac{m\pi}{2}x\sin\frac{n\pi}{3}y\,dy\,dx\\
&= \frac{2}{3}\int_0^2 x(2-x)\sin\frac{m\pi}{2}x\,dx\int_0^3 y(3-y)\sin\frac{n\pi}{3}y\,dy\\
&= \frac{2}{3}\left(\frac{16(1+(-1)^{m+1})}{\pi^3 m^3}\right)\left(\frac{54(1+(-1)^{n+1})}{\pi^3 n^3}\right)\\
&= \frac{576}{\pi^6}\frac{(1+(-1)^{m+1})(1+(-1)^{n+1})}{m^3 n^3}.
\end{aligned}$$

The coefficients λ_{mn} are given by

$$\lambda_{mn} = 6\pi\sqrt{\frac{m^2}{4} + \frac{n^2}{9}} = \pi\sqrt{9m^2 + 4n^2}.$$

Assembling all of these pieces yields

$$\boxed{u(x,y,t) = \frac{576}{\pi^6}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{(1+(-1)^{m+1})(1+(-1)^{n+1})}{m^3n^3}\sin\frac{m\pi}{2}x \times \sin\frac{n\pi}{3}y\cos\pi\sqrt{9m^2+4n^2}\,t\right).}$$

Example 2

Example

Suppose in the previous example we also impose an initial velocity given by

$$g(x, y) = \sin 2\pi x.$$

Find an expression that gives the shape of the membrane for $t > 0$.

- Because B_{mn} depends only on the initial shape, which hasn't changed, we don't need to recompute them.
- We only need to find B^*_{mn} and add the appropriate terms to the previous solution.

Using the values of λ_{mn} computed above, we have

$$\begin{aligned} B_{mn}^* &= \frac{2}{3\pi\sqrt{9m^2+4n^2}} \int_0^2 \int_0^3 \sin 2\pi x \, \sin \frac{m\pi}{2} x \, \sin \frac{n\pi}{3} y \, dy \, dx \\ &= \frac{2}{3\pi\sqrt{9m^2+4n^2}} \int_0^2 sin 2\pi x \sin \frac{m\pi}{2} x \, dx \int_0^3 \sin \frac{n\pi}{3} y \, dy. \end{aligned}$$

The first integral is zero unless $m = 4$, in which case it evaluates to 1. Evaluating the second integral, we have

$$B_{4n}^* = \frac{1}{3\pi\sqrt{36+n^2}} \frac{3(1+(-1)^{n+1})}{n\pi} = \frac{1+(-1)^{n+1}}{\pi^2 n\sqrt{36+n^2}}$$

and $B_{mn}^* = 0$ for $m \neq 4$.

Let $u_1(x,y,t)$ denote the solution obtained in the previous example. If we let

$$\boxed{\begin{aligned} u_2(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}^{*} \sin\frac{m\pi}{2}x \sin\frac{n\pi}{3}y \sin\lambda_{mn}t \\ &= \frac{\sin 2\pi x}{\pi^2}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n\sqrt{36+n^2}} \sin\frac{n\pi}{3}y \sin 2\pi\sqrt{36+n^2}\,t \end{aligned}}$$

then the solution to the present problem is

$$\boxed{u(x,y,t) = u_1(x,y,t) + u_2(x,y,t).}$$