The General Dirichlet Problem on a Rectangle

Ryan C. Daileda

Trinity University

Partial Differential Equations
March 27, 2012

Goal:

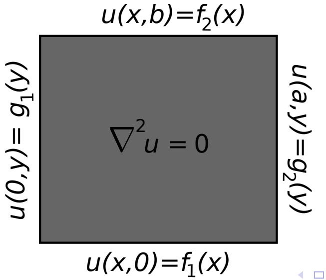
Solve the general (inhomogeneous) Dirichlet problem

$$\nabla^2 u = 0, \qquad 0 < x < a, \qquad 0 < y < b,$$

$$u(x,0) = f_1(x), \quad u(x,b) = f_2(x), \quad 0 \le x \le a,$$

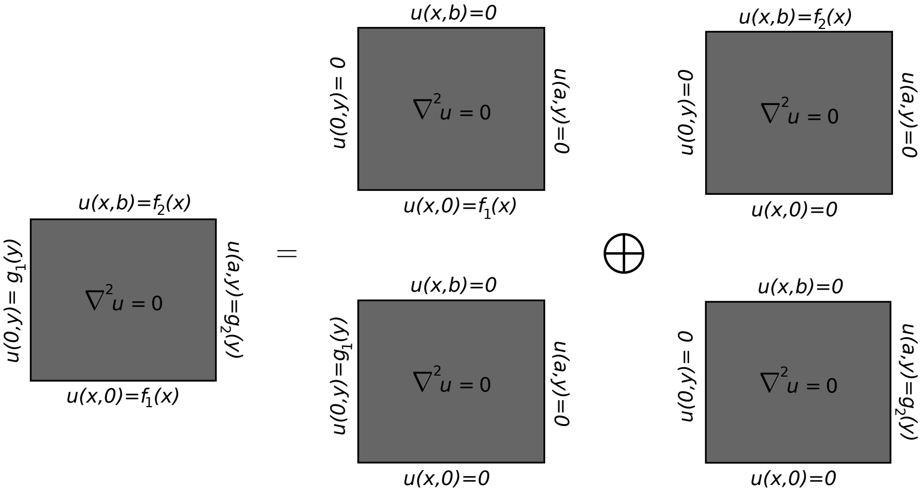
$$u(0,y) = g_1(y), \quad u(a,y) = g_2(y), \quad 0 \le y \le b.$$

Picture:



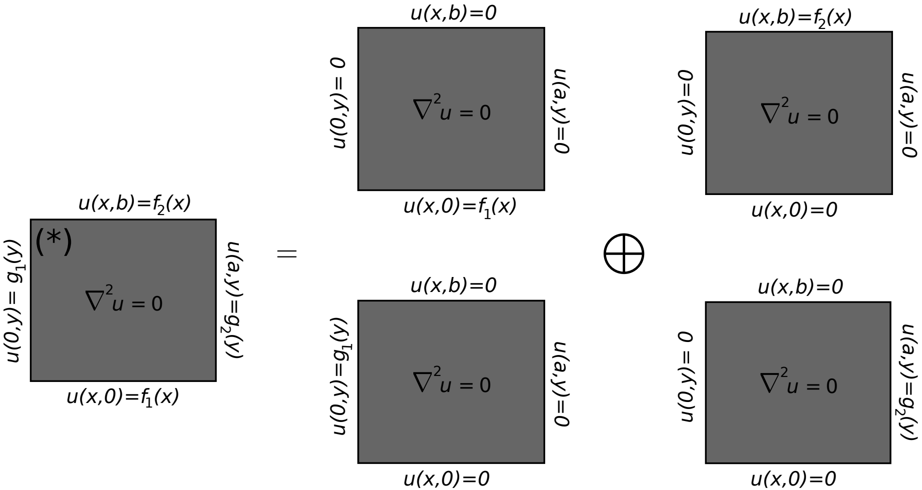
Strategy:

Reduce to four simpler problems and use superposition.



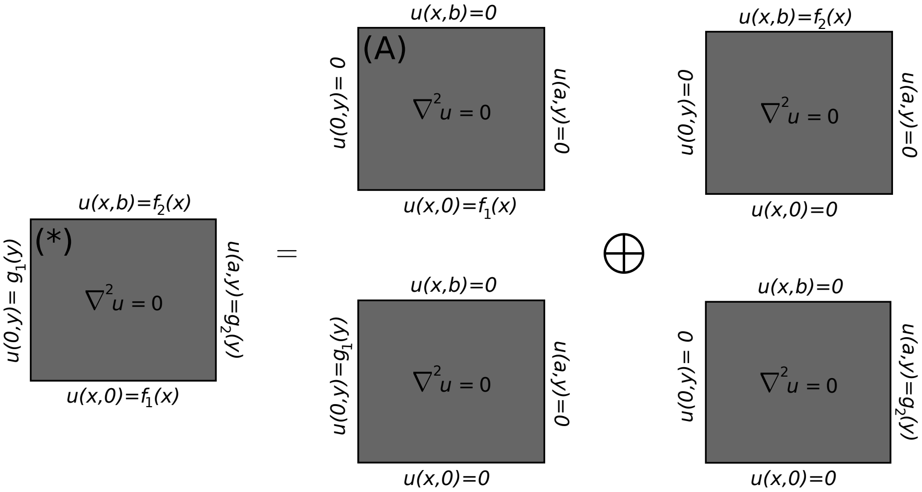
Strategy:

Reduce to four simpler problems and use superposition.



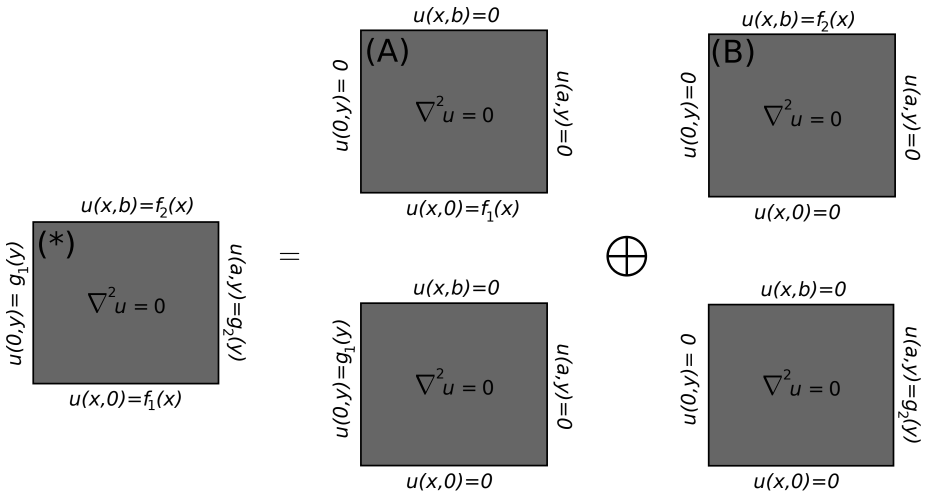
Strategy:

Reduce to four simpler problems and use superposition.



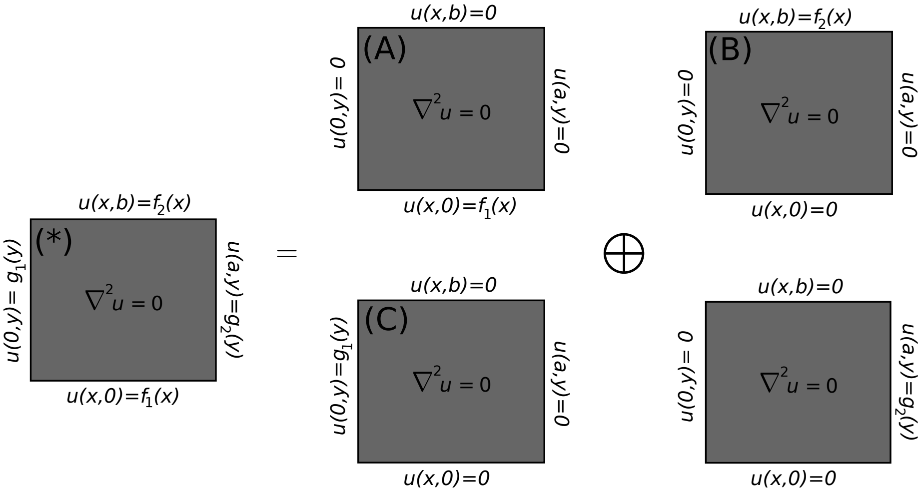
Strategy:

Reduce to four simpler problems and use superposition.



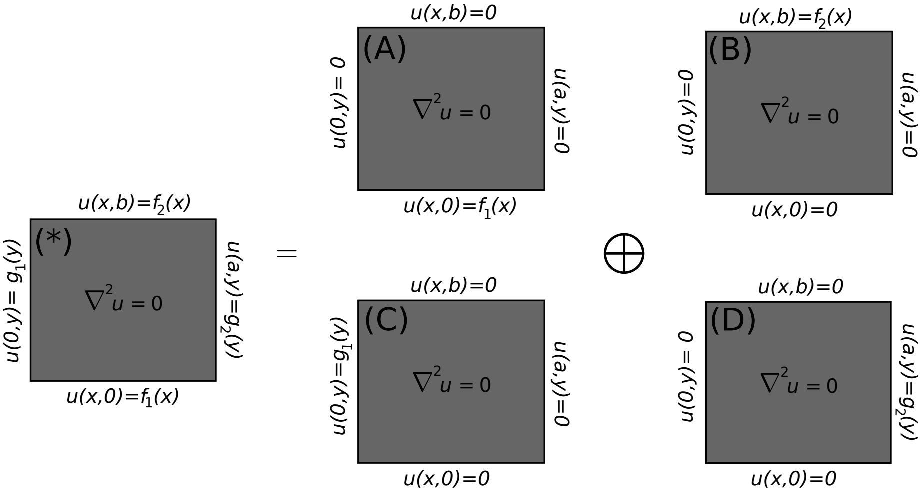
Strategy:

Reduce to four simpler problems and use superposition.



Strategy:

Reduce to four simpler problems and use superposition.



Remarks:

- Explicitly, if u_1, u_2, u_3 and u_4 solve the Dirichlet problems (A), (B), (C) and (D) (respectively), then the general solution to $(*)$ is

$$u = u_1 + u_2 + u_3 + u_4.$$

- Note that the boundary conditions in each of (A) - (D) are homogeneous, with the exception of a single side of the rectangle.
- Problems with more general inhomogeneous boundary conditions (e.g. Neumann or Robin conditions) can be reduced in a similar manner.

Solution to (A) and (B)

We have already seen that the solution to (B) is given by

$$u_2(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\,\sinh\frac{n\pi y}{a},$$

where

$$B_n = \frac{2}{a\sinh\frac{n\pi b}{a}}\int_0^a f_2(x)\sin\frac{n\pi x}{a}\,dx.$$

We can likewise use separation of variables to show that the solution to (A) is

$$u_1(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a}\,\sinh\frac{n\pi(b-y)}{a},$$

where

$$A_n = \frac{2}{a\sinh\frac{n\pi b}{a}}\int_0^a f_1(x)\sin\frac{n\pi x}{a}\,dx.$$

Solution to (C) and (D)

In the same manner we obtain the solution to (C):

$$u_3(x,y) = \sum_{n=1}^{\infty} C_n \sinh\frac{n\pi(a-x)}{b}\,\sin\frac{n\pi y}{b},$$

with

$$C_n = \frac{2}{b\sinh\frac{n\pi a}{b}}\int_0^b g_1(y)\sin\frac{n\pi y}{b}\,dy,$$

as well as the solution to (D):

$$u_4(x,y) = \sum_{n=1}^{\infty} D_n \sinh\frac{n\pi x}{b}\,\sin\frac{n\pi y}{b},$$

where

$$D_n = \frac{2}{b\sinh\frac{n\pi a}{b}}\int_0^b g_2(y)\sin\frac{n\pi y}{b}\,dy.$$

Remarks:

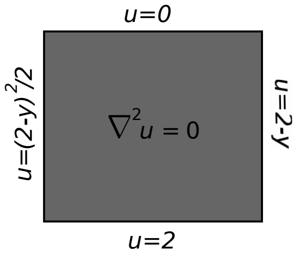
- In each case, the coefficients of the solution are just multiples of the Fourier sine coefficients of the function giving the nonzero boundary condition, e.g.

$$D_n = \frac{1}{\sinh \frac{n\pi a}{b}} \left(n\text{th sine coefficient of } g_2 \text{ on } [0, b] \right)$$

- The coefficients for each boundary condition are independent of the others.
- If any of the boundary conditions is zero, we may omit that term from the solution. E.g. if $g_1 \equiv 0$, then we don't need to include u_3.

Example

Solve the Dirichlet problem on $[0,1]\times[0,2]$ with the following boundary conditions.



We have $a = 1$, $b = 2$ and

$$f_1(x) = 2, \quad f_2(x) = 0, \quad g_1(y) = \frac{(2-y)^2}{2}, \quad g_2(y) = 2 - y.$$

It follows that $B_n = 0$ for all n, and the remaining coefficients we need are

$$A_n = \frac{2}{1 \cdot \sinh \frac{n\pi 2}{1}} \int_0^1 2 \sin \frac{n\pi x}{1}\, dx = \frac{4(1 + (-1)^{n+1})}{n\pi \sinh 2n\pi},$$

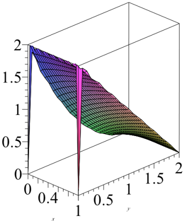
$$C_n = \frac{2}{2 \sinh \frac{n\pi 1}{2}} \int_0^2 \frac{(2-y)^2}{2} \sin \frac{n\pi y}{2}\, dy = \frac{4(\pi^2 n^2 - 2 + 2(-1)^n)}{n^3 \pi^3 \sinh \frac{n\pi}{2}},$$

$$D_n = \frac{2}{2 \sinh \frac{n\pi}{2}} \int_0^2 (2-y) \sin \frac{n\pi y}{2}\, dy = \frac{4}{n\pi \sinh \frac{n\pi}{2}}.$$

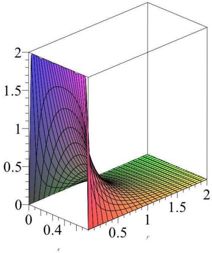
The complete solution is thus

$$
\begin{aligned}
u &= \sum_{n=1}^{\infty} \frac{4(1+(-1)^{n+1})}{n\pi \sinh 2n\pi} \sin n\pi x \sinh n\pi(2-y) \\
&\quad + \sum_{n=1}^{\infty} \frac{4(n^2\pi^2 - 2 + 2(-1)^n)}{n^3\pi^3 \sinh \frac{n\pi}{2}} \sinh \frac{n\pi(1-x)}{2} \sin \frac{n\pi y}{2} \\
&\quad + \sum_{n=1}^{\infty} \frac{4}{n\pi \sinh \frac{n\pi}{2}} \sinh \frac{n\pi x}{2} \sin \frac{n\pi y}{2}.
\end{aligned}
$$

Graphically:



$=$



$\oplus$