The Circular Membrane Problem

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Partial Differential Equations
March 29, 2012

Recall:

The shape of an ideal vibrating thin elastic membrane stretched over a circular frame of radius a can be modeled by

$$u_{tt} = c^2 \nabla^2 u, \qquad x^2 + y^2 < a^2,$$
$$u(x,y,t) = 0, \qquad x^2 + y^2 = a^2.$$

Last time, we saw that in polar coordinates this takes the form

$$u_{tt} = c^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right), \qquad 0 \le r < a, \quad t > 0, \tag{1}$$
$$u(a,\theta,t) = 0, \qquad 0 \le \theta \le 2\pi, \tag{2}$$
$$u(r,0,t) = u(r,2\pi,t),$$
$$u_\theta(r,0,t) = u_\theta(r,2\pi,t), \qquad 0 \le r \le a, \quad t \ge 0. \tag{3}$$

We will also impose the **initial conditions**

$$u(r,\theta,0) = f(r,\theta), \qquad 0 \le r \le a,\ 0 \le \theta \le 2\pi, \qquad (4)$$
$$u_t(r,\theta,0) = g(r,\theta), \qquad 0 \le r \le a,\ 0 \le \theta \le 2\pi, \qquad (5)$$

which give the **initial shape** and **initial velocity** of the membrane, respectively.

Our eventual goal is to completely solve this problem in the usual manner:

- First use separation of variables to find the simplest solutions to (1) - (3);
- Then use superposition to build series solutions that satisfy (4) and (5) as well.

Separation of Variables

Setting $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$ leads to the separated boundary value problems

$$r^2R'' + rR' + \left(\lambda^2 r^2 - \mu^2\right) R = 0, \qquad R(a) = 0,$$

$$\Theta'' + \mu^2\Theta = 0, \qquad \Theta(0) = \Theta(2\pi),$$

$$\Theta'(0) = \Theta'(2\pi),$$

$$T'' + c^2\lambda^2 T = 0.$$

We find immediately that

$$\Theta(\theta) = \Theta_m(\theta) = A_m \cos m\theta + B_m \sin m\theta, \quad \mu = m = 0, 1, 2, \ldots,$$

and that $T(t)$ is a linear combination of $\cos c\lambda t$ and $\sin c\lambda t$.

To determine R and λ, it remains to solve the boundary value problem

$$r^2R'' + rR' + \left(\lambda^2 r^2 - m^2\right) R = 0, \quad (6)$$
$$R(a) = 0. \quad (7)$$

The ODE (6) is the **parametric form of Bessel's equation of order** m. As we will see, it's general solution is given by

$$R(r) = c_1 J_m(\lambda r) + c_2 Y_m(\lambda r)$$

where J_m and Y_m are the **Bessel functions of order** m **of the first** and **second kind**, respectively.

In order to determine λ, c_1 and c_2 so that (7) holds, we need to study these functions.

Bessel's equation

Given $p \geq 0$, the ordinary differential equation

$$x^2 y'' + xy' + (x^2 - p^2)y = 0, \qquad x > 0 \tag{8}$$

is known as **Bessel's equation of order** p.

- Solutions to (8) are known as **Bessel functions.**
- Since (8) is a second order homogeneous linear equation, the general solution is a linear combination of any two linearly independent (fundamental) solutions.
- Our goal is to describe and give the basic properties of the most commonly used pair of fundamental solutions.

Bessel functions of the first kind

The point $x = 0$ is a regular singular point of (8), and the **method of Frobenius** can be used to produce the solution

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p},$$

known as the **Bessel function of order p of the first kind**. Here

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1}\, dt \qquad (x > 0)$$

is the **gamma function**, also called the **generalized factorial function**, since it satisfies

$$\Gamma(x+1) = x\,\Gamma(x) \qquad \text{(the functional equation)},$$
$$\Gamma(n+1) = n! \qquad \text{for } n \in \mathbb{N}.$$

Remarks

- The Bessel functions of the first kind are **special functions**, analogous to sine and cosine.
- Many computer algebra systems include routines for manipulation and evaluation of Bessel functions of the first kind.
- In Maple, the function $J_p(x)$ is invoked by the command

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BesselJ(p,x).
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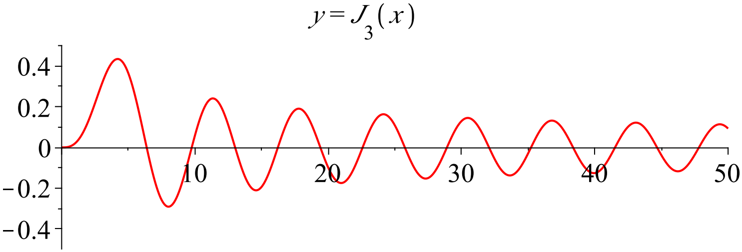
- For some values of p, the Bessel functions of the first kind can be expressed in terms of familiar functions, e.g.

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\left(\frac{3}{x^2} - 1 \right) \sin x - \frac{3}{x} \cos x \right].$$

Properties of Bessel functions of the first kind



- J_p has infinitely many positive zeros, which we denote by

$$0 < \alpha_{p1} < \alpha_{p2} < \alpha_{p3} < \cdots$$

- J_p is oscillatory and tends to zero as $x \to \infty$. More precisely,

$$J_p(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{p\pi}{2} - \frac{\pi}{4}\right)$$

Bessel functions of the second kind

The function J_p provides *one* solution to

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \qquad x > 0.$$

To find the general solution, we need a second linearly independent solution.

- This can be found via the method of **reduction of order**.
- The (appropriately normalized) second solution is denoted by

  $$Y_p(x),$$

  and is called the **Bessel function of order p of the second kind.**

- As with J_p, it is possible to write down explicit series representations of Y_p. We won't need these.
- Many computer algebra systems include routines for manipulation and evaluation of Bessel functions of the second kind.
- In Maple, the function $Y_p(x)$ is invoked by the command

  `BesselY(p,x).`

- For us, the most relevant property of Y_p is

$$\lim_{x \to 0^+} Y_p(x) = -\infty.$$

Back to the vibrating circular membrane

Recall that the radial part $R(r)$ of the separated solution to the vibrating circular membrane problem must satisfy

$$r^2 R'' + rR' + \left(\lambda^2 r^2 - m^2\right) R = 0 \quad (m = 0, 1, 2 \ldots),$$

and that the general solution to this ODE is

$$R(r) = c_1 J_m(\lambda r) + c_2 Y_m(\lambda r).$$

- Because the displacement of the membrane must be finite we require:

$$\lim_{r \to 0^+} R(r) \text{ is finite.}$$

- Since the Bessel functions of the second kind are not finite at zero, we conclude that $c_2 = 0$. Hence, up to a constant

$$R(r) = J_m(\lambda r).$$

If we now impose the boundary condition

$$R(a) = 0,$$

we get

$$J_m(\lambda a) = 0.$$

This means that

$$\lambda a = \alpha_{mn} \qquad \text{or} \qquad \lambda = \lambda_{mn} = \frac{\alpha_{mn}}{a},$$

where α_{mn} is the nth positive zero of J_m. Hence

$$R(r) = R_{mn}(r) = J_m\left(\lambda_{mn} r\right)$$

for any $m = 0, 1, 2, \ldots$ and $n = 1, 2, 3, \ldots$

Remarks

- The zeros α_{mn} are not given by a simple formula, and must typically be computed numerically.
- The functions $R_{mn}(r)$ are the polar analogs of

$$X_m(x) = \sin \frac{m\pi}{a} x$$

  which arose in the rectangular case.
- The numbers $\lambda_{mn} = \alpha_{mn}/a$ are analogous to

$$\mu_m = \frac{m\pi}{a}.$$

- We have (essentially) replace sine by J_m and the zeros of sine by those of J_m.

Normal modes of the vibrating circular membrane

If we now piece together what we've done so far, we find that the **normal modes of the vibrating circular membrane** can be written as

$$u_{mn}(r,\theta,t) = J_m(\lambda_{mn} r)\left(a_{mn}\cos m\theta + b_{mn}\sin m\theta\right)\cos c\lambda_{mn} t,$$
$$u^*_{mn}(r,\theta,t) = J_m(\lambda_{mn} r)\left(a^*_{mn}\cos m\theta + b^*_{mn}\sin m\theta\right)\sin c\lambda_{mn} t$$

for $m = 0, 1, 2, \ldots,\ n = 1, 2, 3, \ldots,$ where $\lambda_{mn} = \alpha_{mn}/a$ and

$$\alpha_{mn} \text{ is the } n\text{th positive zero of } J_m(x).$$

Note that, up to scaling, rotation and a phase shift in time, these all have the form

$$u(r,\theta,t) = J_m(\lambda_{mn} r)\cos m\theta\,\cos c\lambda_{mn} t$$