The Circular Membrane Problem

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Partial Differential Equations
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Recall:

The shape of an ideal vibrating thin elastic membrane stretched over a circular frame of radius $a$ can be modeled by

$$u_{tt} = c^2 \nabla^2 u, \quad x^2 + y^2 < a^2,$$
$$u(x, y, t) = 0, \quad x^2 + y^2 = a^2.$$

Last time, we saw that in polar coordinates this takes the form

$$u_{tt} = c^2 \left( u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right), \quad 0 \leq r < a, \quad t > 0, \quad (1)$$
$$u(a, \theta, t) = 0, \quad 0 \leq \theta \leq 2\pi, \quad (2)$$
$$u(r, 0, t) = u(r, 2\pi, t),$$
$$u_{\theta}(r, 0, t) = u_{\theta}(r, 2\pi, t), \quad 0 \leq r \leq a, \quad t \geq 0. \quad (3)$$
We will also impose the initial conditions

\[ u(r, \theta, 0) = f(r, \theta), \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi, \quad (4) \]
\[ u_t(r, \theta, 0) = g(r, \theta), \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi, \quad (5) \]

which give the initial shape and initial velocity of the membrane, respectively.

Our eventual goal is to completely solve this problem in the usual manner:

- First use separation of variables to find the simplest solutions to (1) - (3);
- Then use superposition to build series solutions that satisfy (4) and (5) as well.
Separation of Variables

Setting \( u(r, \theta, t) = R(r)\Theta(\theta)T(t) \) leads to the separated boundary value problems

\[
\begin{align*}
& r^2 R'' + rR' + \left( \lambda^2 r^2 - \mu^2 \right) R = 0, \quad R(a) = 0, \\
& \Theta'' + \mu^2 \Theta = 0, \quad \Theta(0) = \Theta(2\pi), \\
& \Theta'(0) = \Theta'(2\pi), \\
& T'' + c^2 \lambda^2 T = 0.
\end{align*}
\]

We find immediately that

\[
\Theta(\theta) = \Theta_m(\theta) = A_m \cos m\theta + B_m \sin m\theta, \quad \mu = m = 0, 1, 2, \ldots,
\]

and that \( T(t) \) is a linear combination of \( \cos c\lambda t \) and \( \sin c\lambda t \).
To determine $R$ and $\lambda$, it remains to solve the boundary value problem

$$r^2 R'' + rR' + (\lambda^2 r^2 - m^2) R = 0, \quad (6)$$

$$R(a) = 0. \quad (7)$$

The ODE (6) is the parametric form of Bessel’s equation of order $m$. As we will see, it’s general solution is given by

$$R(r) = c_1 J_m(\lambda r) + c_2 Y_m(\lambda r)$$

where $J_m$ and $Y_m$ are the Bessel functions of order $m$ of the first and second kind, respectively.

In order to determine $\lambda$, $c_1$ and $c_2$ so that (7) holds, we need to study these functions.
Given $p \geq 0$, the ordinary differential equation

$$x^2 y'' + xy' + (x^2 - p^2)y = 0, \quad x > 0$$

(8)

is known as Bessel’s equation of order $p$.

- Solutions to (8) are known as Bessel functions.

- Since (8) is a second order homogeneous linear equation, the general solution is a linear combination of any two linearly independent (fundamental) solutions.

- Our goal is to describe and give the basic properties of the most commonly used pair of fundamental solutions.
Bessel functions of the first kind

The point \( x = 0 \) is a regular singular point of (8), and the method of Frobenius can be used to produce the solution

\[
J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + p + 1)} \left( \frac{x}{2} \right)^{2k+p},
\]

known as the Bessel function of order \( p \) of the first kind. Here

\[
\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \quad (x > 0)
\]

is the gamma function, also called the generalized factorial function, since it satisfies

\[
\Gamma(x + 1) = x \Gamma(x) \quad \text{(the functional equation)},
\]

\[
\Gamma(n + 1) = n! \quad \text{for } n \in \mathbb{N}.
\]
The Bessel functions of the first kind are **special functions**, analogous to sine and cosine.

Many computer algebra systems include routines for manipulation and evaluation of Bessel functions of the first kind.

In Maple, the function $J_p(x)$ is invoked by the command

$$\text{BesselJ}(p, x).$$

For some values of $p$, the Bessel functions of the first kind can be expressed in terms of familiar functions, e.g.

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[ \left( \frac{3}{x^2} - 1 \right) \sin x - \frac{3}{x} \cos x \right].$$
Properties of Bessel functions of the first kind

- $J_p$ has infinitely many positive zeros, which we denote by
  \[ 0 < \alpha_{p1} < \alpha_{p2} < \alpha_{p3} < \cdots \]

- $J_p$ is oscillatory and tends to zero as $x \to \infty$. More precisely,
  \[ J_p(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{p\pi}{2} - \frac{\pi}{4} \right) \]
The function $J_p$ provides one solution to

$$x^2 y'' + xy' + (x^2 - p^2)y = 0, \quad x > 0.$$  

To find the general solution, we need a second linearly independent solution.

- This can be found via the method of **reduction of order**.
- The (appropriately normalized) second solution is denoted by $Y_p(x)$, and is called the **Bessel function of order $p$ of the second kind**.
As with $J_p$, it is possible to write down explicit series representations of $Y_p$. We won’t need these.

Many computer algebra systems include routines for manipulation and evaluation of Bessel functions of the second kind.

In Maple, the function $Y_p(x)$ is invoked by the command

$$\text{BesselY}(p, x).$$

For us, the most relevant property of $Y_p$ is

$$\lim_{x \to 0^+} Y_p(x) = -\infty.$$
Recall that the radial part $R(r)$ of the separated solution to the vibrating circular membrane problem must satisfy

$$r^2 R'' + r R' + (\lambda^2 r^2 - m^2) R = 0 \quad (m = 0, 1, 2 \ldots),$$

and that the general solution to this ODE is

$$R(r) = c_1 J_m(\lambda r) + c_2 Y_m(\lambda r).$$

- Because the displacement of the membrane must be finite we require:

$$\lim_{r \to 0^+} R(r) \text{ is finite.}$$

- Since the Bessel functions of the second kind are not finite at zero, we conclude that $c_2 = 0$. Hence, up to a constant

$$R(r) = J_m(\lambda r).$$
If we now impose the boundary condition

$$ R(a) = 0, $$

we get

$$ J_m(\lambda a) = 0. $$

This means that

$$ \lambda a = \alpha_{mn} \quad \text{or} \quad \lambda = \lambda_{mn} = \frac{\alpha_{mn}}{a}, $$

where $\alpha_{mn}$ is the $n$th positive zero of $J_m$. Hence

$$ R(r) = R_{mn}(r) = J_m(\lambda_{mn}r) $$

for any $m = 0, 1, 2, \ldots$ and $n = 1, 2, 3, \ldots$
Remarks

- The zeros $\alpha_{mn}$ are not given by a simple formula, and must typically be computed numerically.
- The functions $R_{mn}(r)$ are the polar analogs of
  \[ X_m(x) = \sin \frac{m\pi}{a} x \]
  which arose in the rectangular case.
- The numbers $\lambda_{mn} = \alpha_{mn}/a$ are analogous to
  \[ \mu_m = \frac{m\pi}{a}. \]
- We have (essentially) replace sine by $J_m$ and the zeros of sine by those of $J_m$. 
Normal modes of the vibrating circular membrane

If we now piece together what we’ve done so far, we find that the normal modes of the vibrating circular membrane can be written as

\[ u_{mn}(r, \theta, t) = J_m(\lambda_{mn} r) (a_{mn} \cos m\theta + b_{mn} \sin m\theta) \cos c\lambda_{mn} t, \]
\[ u^*_{mn}(r, \theta, t) = J_m(\lambda_{mn} r) (a^*_{mn} \cos m\theta + b^*_{mn} \sin m\theta) \sin c\lambda_{mn} t \]

for \( m = 0, 1, 2, \ldots \), \( n = 1, 2, 3, \ldots \), where \( \lambda_{mn} = \alpha_{mn} / a \) and \( \alpha_{mn} \) is the \( n \)th positive zero of \( J_m(x) \).

Note that, up to scaling, rotation and a phase shift in time, these all have the form

\[ u(r, \theta, t) = J_m(\lambda_{mn} r) \cos m\theta \cos c\lambda_{mn} t \]