Introduction to Sturm-Liouville Theory

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Partial Differential Equations
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Inner products with weight functions

Suppose that $w(x)$ is a nonnegative function on $[a, b]$. If $f(x)$ and $g(x)$ are real-valued functions on $[a, b]$ we define their **inner product on** $[a, b]$ **with respect to the weight** w to be

$$\boxed{\langle f, g\rangle = \int_a^b f(x)g(x)w(x)\,dx.}$$

We say f and g are **orthogonal on** $[a, b]$ **with respect to the weight** w if

$$\langle f, g\rangle = 0.$$

Remarks:

- The inner product and orthogonality depend on the choice of a, b and w.
- When $w(x) \equiv 1$, these definitions reduce to the "ordinary" ones.

Examples

1. The functions $f_n(x) = \sin(nx)$ $(n = 1, 2, \ldots)$ are pairwise orthogonal on $[0, \pi]$ relative to the weight function $w(x) \equiv 1$.
2. Let J_m be the Bessel function of the first kind of order m, and let α_{mn} denote its nth positive zero. Then the functions $f_n(x) = J_m(\alpha_{mn}x/a)$ are pairwise orthogonal on $[0, a]$ with respect to the weight function $w(x) = x$.
3. The functions

$$f_0(x) = 1, \quad f_1(x) = 2x, \quad f_2(x) = 4x^2 - 1, \quad f_3(x) = 8x^3 - 4x,$$
$$f_4(x) = 16x^4 - 12x^2 + 1, \quad f_5(x) = 32x^5 - 32x^3 + 6x$$

are pairwise orthogonal on $[-1, 1]$ relative to the weight function $w(x) = \sqrt{1 - x^2}$. They are examples of **Chebyshev polynomials of the second kind.**

Series expansions

We have frequently seen the need to express a given function as a linear combination of an orthogonal set of functions. Our fundamental result generalizes to weighted inner products.

Theorem

Suppose that $\{f_1, f_2, f_3, \ldots\}$ *is an orthogonal set of functions on* $[a, b]$ *with respect to the weight function* w*. If* f *is a function on* $[a, b]$ *and*

$$f(x) = \sum_{n=1}^{\infty} a_n f_n(x),$$

then the coefficients a_n *are given by*

$$a_n = \frac{\langle f, f_n \rangle}{\langle f_n, f_n \rangle} = \frac{\int_a^b f(x) f_n(x) w(x)\, dx}{\int_a^b f_n^2(x) w(x)\, dx}.$$

Remarks

- The series expansion above is called a **generalized Fourier series for** f, and a_n are the **generalized Fourier coefficients.**
- It is natural to ask:
  - Where do orthogonal sets of functions come from?
  - To what extent is an orthogonal set **complete**, i.e. which functions f have generalized Fourier series expansions?
- In the context of PDEs, these questions are answered by **Sturm-Liouville Theory.**

Sturm-Liouville equations

A **Sturm-Liouville equation** is a second order linear differential equation that can be written in the form

$$\boxed{(p(x)y')' + (q(x) + \lambda r(x))y = 0.}$$

Such an equation is said to be in **Sturm-Liouville form.**

- Here p, q and r are specific functions, and λ is a parameter.
- Because λ is a parameter, it is frequently replaced by other variables or expressions.
- Many "familiar" ODEs that occur during separation of variables can be put in Sturm-Liouville form.

Example

Show that $y'' + \lambda y = 0$ is a Sturm-Liouville equation.

We simply take $p(x) = r(x) = 1$ and $q(x) = 0$.

Example

Put the parametric Bessel equation

$$x^2y'' + xy' + (\lambda^2x^2 - m^2)y = 0$$

in Sturm-Liouville form.

First we divide by x to get

$$\underbrace{xy'' + y'}_{(xy')'} + \left(\lambda^2 x - \frac{m^2}{x}\right) y = 0.$$

This is in Sturm-Liouville form with

$$p(x) = x, \quad q(x) = -\frac{m^2}{x}, \quad r(x) = x,$$

provided we write the parameter as λ^2.

Example

Put **Legendre's differential equation**

$$y'' - \frac{2x}{1-x^2}y' + \frac{\mu}{1-x^2}y = 0$$

in Sturm-Liouville form.

First we multiply by $1 - x^2$ to get

$$\underbrace{(1-x^2)y'' - 2xy'}_{((1-x^2)y')'} + \mu y = 0.$$

This is in Sturm-Liouville form with

$$p(x) = 1 - x^2, \quad q(x) = 0, \quad r(x) = 1,$$

provided we write the parameter as μ.

Example

Put **Chebyshev's differential equation**

$$(1-x^2)y'' - xy' + n^2 y = 0$$

in Sturm-Liouville form.

First we divide by $\sqrt{1-x^2}$ to get

$$\underbrace{\sqrt{1-x^2}\,y'' - \frac{x}{\sqrt{1-x^2}}\,y'}_{(\sqrt{1-x^2}\,y')'} + \frac{n^2}{\sqrt{1-x^2}}\,y = 0.$$

This is in Sturm-Liouville form with

$$p(x) = \sqrt{1-x^2}, \quad q(x) = 0, \quad r(x) = \frac{1}{\sqrt{1-x^2}},$$

provided we write the parameter as n^2.

Sturm-Liouville problems

A **Sturm-Liouville problem** consists of

- A Sturm-Liouville equation on an interval:

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0, \quad a < x < b, \tag{1}$$

  together with

- **Boundary conditions**, i.e. specified behavior of y at $x = a$ and $x = b$.

We will assume that p, p', q and r are continuous and $p > 0$ on (at least) the open interval $a < x < b$.

According to the general theory of second order linear ODEs, this guarantees that solutions to (1) exist.

Regularity conditions

A **regular Sturm-Liouville problem** has the form

$$
\begin{aligned}
(p(x)y')' + (q(x) + \lambda r(x))y &= 0, \quad a < x < b, \\
c_1 y(a) + c_2 y'(a) &= 0, \qquad (2) \\
d_1 y(b) + d_2 y'(b) &= 0, \qquad (3)
\end{aligned}
$$

where:

- $(c_1, c_2) \neq (0,0)$ and $(d_1, d_2) \neq (0,0)$;
- p, p', q and r are continuous on $[a, b]$;
- p and r are positive on $[a, b]$.

The boundary conditions (2) and (3) are called **separated** boundary conditions.

Example

The boundary value problem

$$\begin{aligned} y'' + \lambda y &= 0, \quad 0 < x < L, \\ y(0) &= y(L) = 0, \end{aligned}$$

is a regular Sturm-Liouville problem (recall that $p(x) = r(x) = 1$ *and* $q(x) = 0$*).*

Example

The boundary value problem

$$\begin{aligned} ((x^2+1)y')' + (x+\lambda)y &= 0, \quad -1 < x < 1, \\ y(-1) &= y'(1) = 0, \end{aligned}$$

is a regular Sturm-Liouville problem (here $p(x) = x^2 + 1$, $q(x) = x$ *and* $r(x) = 1$*).*

Example

The boundary value problem

$$\begin{aligned} x^2y'' + xy' + (\lambda^2x^2 - m^2)y &= 0, \quad 0 < x < a, \\ y(a) &= 0, \end{aligned}$$

is **not** *a regular Sturm-Liouville problem.*

Why not? Recall that when put in Sturm-Liouville form we had $p(x) = r(x) = x$ and $q(x) = -m^2/x$. There are several problems:

- p and r are **not positive** when $x = 0$.
- q is **not continuous** when $x = 0$.
- The boundary condition at $x = 0$ is **missing**.

This is an example of a **singular Sturm-Liouville problem.**

Eigenvalues and eigenfunctions

A **nonzero** function y that solves the Sturm-Liouville problem

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0, \quad a < x < b,$$
$$\text{(plus boundary conditions)},$$

is called an **eigenfunction**, and the corresponding value of λ is called its **eigenvalue**.

- The **eigenvalues** of a Sturm-Liouville problem are the values of λ for which nonzero solutions exist.
- We can talk about eigenvalues and eigenfunctions for regular or singular problems.

Example

Find the eigenvalues of the regular Sturm-Liouville problem

$$\begin{aligned} y'' + \lambda y &= 0, \qquad 0 < x < L, \\ y(0) &= y(L) = 0, \end{aligned}$$

This problem first arose when separated variables in the 1-D wave equation. We already know that nonzero solutions occur only when

$$\boxed{\lambda = \lambda_n = \frac{n^2\pi^2}{L^2} \qquad \text{(eigenvalues)}}$$

and

$$\boxed{y = y_n = \sin\frac{n\pi x}{L} \qquad \text{(eigenfunctions)}}$$

for $n = 1, 2, 3, \ldots$

Example

Find the eigenvalues of the regular Sturm-Liouville problem

$$\begin{aligned} y'' + \lambda y &= 0, \quad 0 < x < L, \\ y(0) &= 0, \quad y(L) + y'(L) = 0, \end{aligned}$$

This problem arose when we separated variables in the 1-D heat equation with Robin conditions. We already know that nonzero solutions occur only when

$$\boxed{\lambda = \lambda_n = \mu_n^2,}$$

where μ_n is the nth positive solution to

$$\tan \mu L = -\mu,$$

and

$$\boxed{y = y_n = \sin(\mu_n x)}$$

for $n = 1, 2, 3, \ldots$

Example

If $m \geq 0$, find the eigenvalues of the singular Sturm-Liouville problem

$$\begin{aligned} x^2y'' + xy' + (\lambda^2x^2 - m^2)y &= 0, \qquad 0 < x < a, \\ y(0) \text{ is finite}, &\qquad y(a) = 0. \end{aligned}$$

This problem arose when we separated variables in the vibrating circular membrane problem. We know that nonzero solutions occur only when

$$\boxed{\lambda = \lambda_n = \frac{\alpha_{mn}}{a},}$$

where α_{mn} is the nth positive zero of the Bessel function J_m, and

$$\boxed{y = y_n = J_m(\lambda_n x)}$$

for $n = 1, 2, 3, \ldots$ (technically, the eigenvalues are $\lambda_n^2 = \alpha_{mn}^2/a^2$.)

The previous examples demonstrate the following general properties of a regular Sturm-Liouville problem

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0, \qquad a < x < b,$$
$$c_1 y(a) + c_2 y'(a) = 0, \qquad d_1 y(b) + d_2 y'(b) = 0.$$

Theorem

The eigenvalues form an increasing sequence of real numbers

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

with

$$\lim_{n\to\infty} \lambda_n = \infty.$$

Moreover, the eigenfunction y_n corresponding to λ_n is unique (up to a scalar multiple), and has exactly $n - 1$ zeros in the interval $a < x < b$.

Another general property is the following.

Theorem

Suppose that y_j and y_k are eigenfunctions corresponding to distinct eigenvalues λ_j and λ_k. Then y_j and y_k are orthogonal on $[a, b]$ with respect to the weight function $w(x) = r(x)$. That is

$$\langle y_j, y_k \rangle = \int_a^b y_j(x) y_k(x) r(x)\, dx = 0.$$

- This theorem actually holds for certain non-regular Sturm-Liouville problems, such as those involving Bessel's equation.
- Applying this result in the examples above we immediately recover familiar orthogonality statements.
- This result explains **why** orthogonality figures so prominently in all of our work.

Examples

Example

Write down the conclusion of the orthogonality theorem for

$$\begin{aligned} y'' + \lambda y &= 0, \qquad 0 < x < L, \\ y(0) &= y(L) = 0. \end{aligned}$$

Since the eigenfunctions of this regular Sturm-Liouville problem are $y_n = \sin(n\pi x/L)$, and since $r(x) = 1$, we **immediately** deduce that

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right)\, dx = 0$$

for $m \neq n$.

Example

If $m \geq 0$, write down the conclusion of the orthogonality theorem for

$$\begin{aligned} x^2y'' + xy' + (\lambda^2x^2 - m^2)y &= 0, \qquad 0 < x < a, \\ y(0) \text{ is finite,} \qquad & y(a) = 0. \end{aligned}$$

Since the eigenfunctions of this regular Sturm-Liouville problem are $y_n = J_m(\alpha_{mn}x/a)$, and since $r(x) = x$, we **immediately** deduce that

$$\int_0^a J_m\left(\frac{\alpha_{mk}}{a}x\right) J_m\left(\frac{\alpha_{m\ell}}{a}x\right) x\,dx = 0$$

for $k \neq \ell$.