More on Sturm-Liouville Theory

Ryan C. Daileda

Trinity University

Partial Differential Equations
April 19, 2012

Recall:

A **Sturm-Liouville (S-L) problem** consists of

- A Sturm-Liouville equation on an interval:

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0, \quad a < x < b,$$

  together with

- **Boundary conditions**, i.e. specified behavior of y at $x = a$ and $x = b$.

Such a problem is called **regular** if:

- The boundary conditions are of the form

$$\begin{aligned} c_1 y(a) + c_2 y'(a) &= 0, \\ d_1 y(b) + d_2 y'(b) &= 0, \end{aligned}$$

  where $(c_1, c_2), (d_1, d_2) \neq (0, 0)$;

- p, q and r satisfy certain **regularity conditions** on $[a, b]$.

A **nonzero** function y that solves an S-L problem is called an **eigenfunction**, and the corresponding value of λ is called an **eigenvalue**.

Eigenvalues and eigenfunctions of (regular) S-L problems have very nice properties.

Theorem

The eigenvalues of a regular S-L problem form an increasing sequence of real numbers

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

with

$$\lim_{n\to\infty} \lambda_n = \infty.$$

Moreover, the eigenfunction y_n corresponding to λ_n is unique (up to a scalar multiple), and has exactly $n-1$ zeros in the interval $a < x < b$.

Theorem

Suppose that y_j and y_k are eigenfunctions corresponding to distinct eigenvalues λ_j and λ_k of a (regular) S-L problem. Then y_j and y_k are orthogonal on $[a, b]$ with respect to the weight function $w(x) = r(x)$. That is

$$\langle y_j, y_k \rangle = \int_a^b y_j(x) y_k(x) r(x)\, dx = 0.$$

- We have put the word "regular" in parentheses because this result actually holds for certain non-regular S-L problems, too.
- We will look at the proof of this result to see just where "regularity" is needed.

Proof of orthogonality

If (y_j, λ_j), (y_k, λ_k) are eigenfunction/eigenvalue pairs then

$$\begin{aligned} (py_j')' + (q + \lambda_j r)y_j &= 0, \\ (py_k')' + (q + \lambda_k r)y_k &= 0. \end{aligned}$$

Multiply the first by y_k and the second by y_j, then subtract to get

$$(py_j')'y_k - (py_k')'y_j + (\lambda_j - \lambda_k)y_j y_k r = 0.$$

Moving the λ-terms to one side and "adding zero," we get

$$\begin{aligned} (\lambda_j - \lambda_k)y_j y_k r &= (py_k')'y_j - (py_j')'y_k \\ &= (py_k')'y_j + py_k'y_j' - py_j'y_k' - (py_j')'y_k \\ &= (py_k'y_j - py_j'y_k)' \\ &= \left(p(y_k'y_j - y_j'y_k)\right)'. \end{aligned}$$

If $\lambda_j \neq \lambda_k$, we can divide by $\lambda_j - \lambda_k$ and then integrate to get

$$\boxed{\langle y_j, y_k \rangle = \int_a^b y_j(x) y_k(x) r(x)\, dx = \left. \frac{p(x)\left(y_k'(x) y_j(x) - y_j'(x) y_k(x)\right)}{\lambda_j - \lambda_k} \right|_a^b .}$$

This proves the orthogonality of y_j and y_k whenever the RHS equals zero. This is guaranteed to happen if

$$p(a)\left(y_k'(a) y_j(a) - y_j'(a) y_k(a)\right) = p(b)\left(y_k'(b) y_j(b) - y_j'(b) y_k(b)\right) = 0.$$

These equalities occur when:

- $\underbrace{y_k'(a) y_j(a) - y_j'(a) y_k(a) = 0}_{A}$ or $\underbrace{p(a) = 0}_{A'}$;
- $\underbrace{y_k'(b) y_j(b) - y_j'(b) y_k(b) = 0}_{B}$ or $\underbrace{p(b) = 0}_{B'}$.

While these conditions are sufficient for orthogonality, it should be pointed out that they are not necessary.

Orthogonality for regular S-L problems

If our S-L problem is regular then at $x = a$ we have

$$\begin{aligned} c_1 y_j(a) + c_2 y_j'(a) &= 0, \\ c_1 y_k(a) + c_2 y_k'(a) &= 0, \end{aligned}$$

or in matrix form

$$\begin{pmatrix} y_j(a) & y_j'(a) \\ y_k(a) & y_k'(a) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $(c_1, c_2) \neq (0, 0)$ the determinant must be zero, that is

$$y_j(a) y_k'(a) - y_k(a) y_j'(a) = 0,$$

which is condition A. Likewise, the boundary condition at $x = b$ gives condition B, which verifies orthogonality.

Examples

Example

Use the preceding results to verify orthogonality of the eigenfunctions of

$$\begin{aligned} y'' + \lambda y &= 0, \quad 0 < x < L, \\ y(0) &= y(L) = 0. \end{aligned}$$

This is a **regular** S-L problem with eigenfunctions

$$y_n = \sin(n\pi x/L).$$

Since $r(x) = 1$, we **immediately** deduce that

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right)\, dx = 0$$

for $m \neq n$.

Example

If $m \geq 0$, use the preceding results to verify orthogonality of the eigenfunctions of

$$\begin{aligned} x^2y'' + xy' + (\lambda^2x^2 - m^2)y &= 0, \qquad 0 < x < a, \\ y(0) \ \textit{is finite}, \qquad & y(a) = 0. \end{aligned}$$

This is a **singular** S-L problem with eigenfunctions

$$y_n = J_m(\alpha_{mn}x/a).$$

Since $p(x) = x$, $p(0) = 0$. This gives condition A'. Since the boundary condition $y(a) = 0$ is regular, we get also get condition B. With $r(x) = x$, we **immediately** deduce that

$$\int_0^a J_m\left(\frac{\alpha_{mk}}{a}x\right) J_m\left(\frac{\alpha_{m\ell}}{a}x\right) x\,dx = 0$$

for $k \neq \ell$.

Example

Use the preceding results to verify orthogonality of the eigenfunctions of

$$\begin{aligned} y'' + \lambda y &= 0, \quad -p < x < p, \\ y(-p) &= y(p), \\ y'(-p) &= y'(p). \end{aligned}$$

This is an S-L problem with $2p$-**periodic boundary conditions**. It is left as an exercise to verify that the eigenvalues are

$$\lambda_n = \frac{n^2\pi^2}{p^2}$$

for $n = 0, 1, 2, 3, \ldots$ with eigenfunctions

$$y_n = \cos\frac{n\pi x}{p} \text{ or } \sin\frac{n\pi x}{p}.$$

Although A, A', B and B' may not hold, the periodic boundary conditions imply that

$$\left(y_k'(p)y_j(p) - y_j'(p)y_k(p)\right) - \left(y_k'(-p)y_j(-p) - y_j'(-p)y_k(-p)\right) = 0.$$

Since $r(x) = 1$, this **immediately** implies the orthogonality relations

$$\begin{aligned}
\int_{-p}^{p} \sin\frac{m\pi x}{p}\,\sin\frac{n\pi x}{p}\,dx &= 0,\\
\int_{-p}^{p} \cos\frac{m\pi x}{p}\,\cos\frac{n\pi x}{p}\,dx &= 0,\\
\int_{-p}^{p} \sin\frac{m\pi x}{p}\,\cos\frac{n\pi x}{p}\,dx &= 0,
\end{aligned}$$

for $m \neq n$.

"Fourier convergence" for S-L problems

The eigenfunctions of an S-L problem provide a family of orthogonal functions. As with sine and cosine, we can use these to give series expansions for "sufficiently nice" functions.

Theorem

Let $y_1, y_2, y_3, \ldots$ be the eigenfunctions of a **regular** *S-L problem on $[a, b]$. If f is piecewise smooth on $[a, b]$, then*

$$\frac{f(x+) + f(x-)}{2} = \sum_{n=1}^{\infty} A_n y_n(x),$$

where

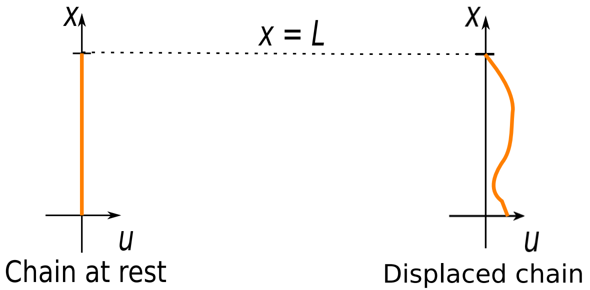
$$A_n = \frac{\langle f, y_n \rangle}{\langle y_n, y_n \rangle} = \frac{\displaystyle\int_a^b f(x) y_n(x) r(x)\, dx}{\displaystyle\int_a^b y_n^2(x) r(x)\, dx}.$$

Remarks

- The series $\sum_{n=1}^{\infty} A_n y_n$ is called the **eigenfunction expansion** of f.
- Recall that $f(x) = \frac{f(x+)+f(x-)}{2}$ anywhere f is continuous. So the eigenfunction expansion is equal to f at most points.
- Although we have only stated this result for regular S-L problems, it frequently holds for singular problems as well.
- The "original" Fourier convergence theorem provides an example of this phenomenon (the S-L problem involved in this case is non-regular).

The hanging chain

Consider a chain (or heavy rope, cable, etc.) of length L hanging from a fixed point, subject to only to downward gravitational force.



We place the chain along the (vertical) x-axis, displace the chain from rest, and let

$u(x,t) =$ Horizontal deflection of chain from equilibrium at height x and time t.

Under ideal assumptions (e.g. planar motion, small deflection, no energy loss due to friction or air resistance, etc.) we obtain the boundary value problem

$$\begin{aligned} u_{tt} &= g\left(xu_{xx} + u_x\right), && 0 < x < L, \quad t > 0, \\ u(L,t) &= 0, && t > 0, \\ u(x,0) &= f(x), && \\ u_t(x,0) &= v(x), && \end{aligned}$$

where

- $f(x)$ is the **initial shape** of the chain,
- $v(x)$ is the **initial (horizontal) velocity** of the chain,
- g is the acceleration due to gravity.

Writing $u(x,t) = X(x)T(t)$, separation of variables (and physical considerations) yields

$$\begin{aligned} T'' + \lambda^2 g T &= 0, \qquad t > 0, \\ xX'' + X' + \lambda^2 X &= 0, \qquad 0 < x < L, \\ X(0) \text{ finite, } X(L) &= 0. \end{aligned}$$

The general solution for T is

$$T(t) = A\cos\left(\sqrt{g}\lambda t\right) + B\sin\left(\sqrt{g}\lambda t\right).$$

The ODE for X can be rewritten as

$$(xX')' + \lambda^2 X = 0,$$

yielding a **singular** S-L problem (with $p(x) = x$, $q(x) = 0$, $r(x) = 1$, and parameter λ^2).

To find the eigenfunctions, we substitute $s = 2\sqrt{x}$. This yields the parametric Bessel equation of order 0:

$$s^2 \frac{d^2X}{ds^2} + s\frac{dX}{ds} + \lambda^2 s^2 X = 0, \qquad 0 < s < 2\sqrt{L},$$
$$X(0) \text{ finite}, \; X(2\sqrt{L}) = 0.$$

As we have seen, this means

$$\lambda = \lambda_n = \frac{\alpha_n}{2\sqrt{L}}$$
$$X(s) = X_n(s) = J_0\left(\frac{\alpha_n s}{2\sqrt{L}}\right),$$

where α_n is the nth positive zero of J_0. Back-substitution then gives

$$X(x) = X_n(x) = J_0\left(\alpha_n \sqrt{\frac{x}{L}}\right).$$

From this we find that

$$T(t) = T_n(t) = A_n \cos\left(\sqrt{g}\lambda_n t\right) + B_n \sin\left(\sqrt{g}\lambda_n t\right)$$
$$= A_n \cos\left(\sqrt{\frac{g}{L}}\frac{\alpha_n t}{2}\right) + B_n \sin\left(\sqrt{\frac{g}{L}}\frac{\alpha_n t}{2}\right),$$

and superposition gives the **general solution**

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$$
$$= \sum_{n=1}^{\infty} J_0\left(\alpha_n \sqrt{\frac{x}{L}}\right)\left(A_n \cos\left(\sqrt{\frac{g}{L}}\frac{\alpha_n t}{2}\right) + B_n \sin\left(\sqrt{\frac{g}{L}}\frac{\alpha_n t}{2}\right)\right).$$

The initial shape condition requires that

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \underbrace{J_0\left(\alpha_n \sqrt{\frac{x}{L}}\right)}_{X_n(x)}.$$

According to S-L theory, this means that

$$\boxed{\begin{aligned} A_n = \frac{\langle f, X_n\rangle}{\langle X_n, X_n\rangle} &= \frac{\int_0^L f(x) J_0\left(\alpha_n\sqrt{\frac{x}{L}}\right)\,dx}{\int_0^L J_0^2\left(\alpha_n\sqrt{\frac{x}{L}}\right)\,dx} \\ &= \frac{1}{L\,J_1^2(\alpha_n)}\int_0^L f(x) J_0\left(\alpha_n\sqrt{\frac{x}{L}}\right)\,dx. \end{aligned}}$$

Setting $u_t(x,0) = v(x)$ and using similar reasoning yields

$$\boxed{B_n = \frac{2}{\alpha_n J_1^2(\alpha_n)\sqrt{gL}}\int_0^L v(x) J_0\left(\alpha_n\sqrt{\frac{x}{L}}\right)\,dx.}$$