More on the Circular Membrane Problem

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Partial Differential Equations
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Recall:

The vibrations in a thin circular membrane of radius a can be modeled by the boundary value problem

$$
\begin{aligned}
&u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right), && 0 < r < a, \quad t > 0, && (1)\\
&u(a,\theta,t) = 0, && 0 \le \theta \le 2\pi, \quad t > 0 && (2)\\
&u(r,0,t) = u(r,2\pi,t), && && \\
&u_\theta(r,0,t) = u_\theta(r,2\pi,t), && 0 < r < a, \quad t > 0. && (3)
\end{aligned}
$$

Here we have centered the membrane at the origin in the xy-plane, (r,θ) are polar coordinates, and

$$u(r,\theta,t) = \text{deflection of membrane from equilibrium at position } (r,\theta) \text{ and time } t.$$

Normal modes of the vibrating circular membrane

Using separation of variables we found the **normal modes**

$$u_{mn}(r,\theta,t) = J_m(\lambda_{mn} r)\left(a_{mn}\cos m\theta + b_{mn}\sin m\theta\right)\cos c\lambda_{mn} t,$$

$$u^*_{mn}(r,\theta,t) = J_m(\lambda_{mn} r)\left(a^*_{mn}\cos m\theta + b^*_{mn}\sin m\theta\right)\sin c\lambda_{mn} t$$

for $m = 0, 1, 2, \ldots$, $n = 1, 2, 3, \ldots$, where

- J_m is the **Bessel function of order** m **of the first kind**,
- $\lambda_{mn} = \alpha_{mn}/a$, and
- α_{mn} is the nth positive zero of J_m.

We'll briefly review what we know about Bessel functions of the first kind.

Bessel functions of the first kind

Given $p \geq 0$, $J_p(x)$ is a particular solution to the **Bessel equation of order** p

$$\boxed{x^2y'' + xy' + (x^2 - p^2)y = 0, \quad x > 0.}$$

Relevant properties include:

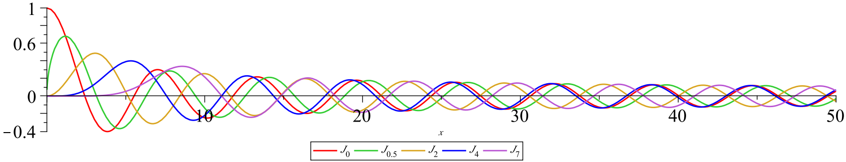
- J_p has infinitely many positive zeros, which we denote by

$$0 < \alpha_{p1} < \alpha_{p2} < \alpha_{p3} < \cdots$$

- $J_0(0) = 1$ and $J_p(0) = 0$ for $p > 0$.
- J_p is oscillatory and tends to zero as $x \to \infty$. More precisely,

$$J_p(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{p\pi}{2} - \frac{\pi}{4}\right).$$

Picture:



- The values of J_p always lie between 1 and -1.
- The distance between successive zeros α_{pn} and $\alpha_{p,n+1}$ of J_p approaches π as $n \to \infty$.
- For $0 < p < 1$, the graph of J_p has a vertical tangent line at $x = 0$.
- For $1 < p$, the graph of J_p has a horizontal tangent line at $x = 0$, and the graph is initially "flat."

The general solution to the vibrating circular membrane problem

Superposition of the normal modes gives the **general solution** to (1) - (3)

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a_{mn}\cos m\theta + b_{mn}\sin m\theta\right)\cos c\lambda_{mn} t$$
$$+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a^*_{mn}\cos m\theta + b^*_{mn}\sin m\theta\right)\sin c\lambda_{mn} t.$$

We now need to determine the values of the coefficients a_{mn}, b_{mn}, a^*_{mn} and b^*_{mn} so that the solution satisfies the **initial conditions**

$$u(r,\theta,0) = f(r,\theta), \qquad \text{(the \textbf{initial shape}),}$$
$$u_t(r,\theta,0) = g(r,\theta), \qquad \text{(the \textbf{initial velocity}).}$$

Setting $t = 0$ in the general solution we find that these conditions require us to satisfy

$$f(r,\theta) = u(r,\theta,0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a_{mn}\cos m\theta + b_{mn}\sin m\theta\right)$$

$$g(r,\theta) = u_t(r,\theta,0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c\lambda_{mn} J_m(\lambda_{mn} r)\left(a_{mn}^{*}\cos m\theta + b_{mn}^{*}\sin m\theta\right)$$

We find that:

- The coefficients a_{mn} and b_{mn} only have to do with the initial shape, and a_{mn}^{*} and b_{mn}^{*} only have to do with the initial velocity.
- The relationship between a_{mn}, b_{mn} and $f(r,\theta)$ is (up to a factor of $c\lambda_{mn}$) the same as the relationship between a_{mn}^{*}, b_{mn}^{*} and $g(r,\theta)$.

Fourier-Bessel expansions

The initial shape equation

$$f(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a_{mn}\cos m\theta + b_{mn}\sin m\theta\right)$$

is called the **Fourier-Bessel expansion** of f. It requires us to write $f(r,\theta)$ as a linear combination of the functions

$$\phi_{mn}(r,\theta) = J_m(\lambda_{mn} r)\cos m\theta \text{ and } \psi_{mn}(r,\theta) = J_m(\lambda_{mn} r)\sin m\theta$$

for $m = 0, 1, 2, \ldots,\ n = 1, 2, 3 \ldots.$

As usual, we can use **orthogonality** to express the coefficients in this combination as ratios of inner products (integrals).

Orthogonality

Theorem

The functions

$$\phi_{mn}(r,\theta) = J_m(\lambda_{mn}r)\cos m\theta \text{ and } \psi_{mn}(r,\theta) = J_m(\lambda_{mn}r)\sin m\theta$$

$(m = 0, 1, 2, \ldots,\ n = 1, 2, 3\ldots)$ form a complete orthogonal set of functions relative to the **inner product**

$$\langle f, g\rangle = \int_0^{2\pi}\int_0^a f(r,\theta)g(r,\theta)r\,dr\,d\theta.$$

That is,

$$\langle \phi_{mn}, \phi_{jk}\rangle = \langle \psi_{mn}, \psi_{jk}\rangle = 0 \qquad \text{for } (m,n) \neq (j,k),$$
$$\langle \phi_{mn}, \psi_{jk}\rangle = 0 \qquad \text{for all } (m,n) \text{ and } (j,k).$$

Proving orthogonality

We will prove the orthogonality of the functions

$$\phi_{mn}(r,\theta) = J(\lambda_{mn} r)\cos m\theta.$$

The other cases will be left as exercises. If $(m,n) \neq (j,k)$, we have

$$\begin{aligned}\langle \phi_{mn}, \phi_{jk} \rangle &= \int_0^{2\pi}\int_0^a J_m(\lambda_{mn} r)\,\cos(m\theta)\,J_j(\lambda_{jk} r)\,\cos(j\theta) r\,dr\,d\theta \\ &= \int_0^{2\pi} \cos(m\theta)\,\cos(j\theta)\,d\theta \int_0^a J_m(\lambda_{mn} r)\,J_j(\lambda_{jk} r)\,r\,dr.\end{aligned}$$

By orthogonality of the functions $\{\cos m\theta\}$ on $[0, 2\pi]$, the first integral is **zero** if $m \neq j$. What if $m = j$?

If $m = j$ then the first integral is **not** zero. So to establish orthogonality in this case we must show that

$$\boxed{\int_0^a J_m(\lambda_{mn} r)\, J_m(\lambda_{mk} r)\, r\, dr = 0 \quad \text{if } n \neq k.}$$

- This says that the functions $J_m(\lambda_{mn} r)$, $m = 0, 1, 2 \ldots$, are orthogonal on the interval $[0, a]$ relative to the inner product

$$\langle f, g \rangle = \int_0^a f(r) g(r) r\, dr.$$

- Note the presence of the function $w(r) = r$ in this inner product. It is called a **weight function**.
- Inner products with weight functions figure prominently in **Sturm-Liouville theory**.

Orthogonality of Bessel functions

The function $y = J_m(\lambda r)$ satisfies the ODE

$$r^2 y'' + ry' + (\lambda^2 r^2 - m^2)y = 0.$$

This is equivalent to:

$$r(ry')' = -(\lambda^2 r^2 - m^2)y.$$

Taking $y_n = J_m(\lambda_{mn} r)$ and $y_k = J_m(\lambda_{mk} r)$ (for convenience) gives

$$r(ry_n')' = -(\lambda_{mn}^2 r^2 - m^2)y_n$$
$$r(ry_k')' = -(\lambda_{mk}^2 r^2 - m^2)y_k.$$

Multiply the first by y_k, the second by y_n, and subtract to get

$$(\lambda_{mk}^2 - \lambda_{mn}^2)y_n y_k r^2 = ry_k(ry_n')' - ry_n(ry_k')'.$$

Now divide by r:

$$\begin{aligned}(\lambda_{mk}^2 - \lambda_{mn}^2) y_n y_k r &= y_k (r y_n')' - y_n (r y_k')' \\ &= y_k (r y_n')' + y_k' r y_n' - y_n' r y_k' - y_n (r y_k')' \\ &= (y_k r y_n' - y_n r y_k')'.\end{aligned}$$

Now integrate both sides:

$$(\lambda_{mk}^2 - \lambda_{mn}^2) \int_0^a y_n y_k r \, dr = y_k r y_n' - y_n r y_k' \Big|_0^a. \tag{4}$$

Finally, we have

$$y_n(a) = J_m(\lambda_{mn} a) = J_m\left(\frac{\alpha_{mn}}{a} a\right) = J_m(\alpha_{mn}) = 0$$

and likewise $y_k(a) = 0$. So, the right hand side of (4) equals **zero**.

Since $\lambda_{mk} \neq \lambda_{mn}$ for $k \neq n$, it must be the case that

$$0 = \int_0^a y_n y_k r\, dr = \int_0^a J_m(\lambda_{mn} r) J_m(\lambda_{mk} r) r\, dr$$

which is what we needed to show.

- In the case of the inner product of $J_m(\lambda_{mn} r)$ with itself, we have the complementary relation

$$\boxed{\int_0^a J_m^2(\lambda_{mn} r) r\, dr = \frac{a^2}{2} J_{m+1}^2(\alpha_{mn})}$$

  for $n = 1, 2, 3, \ldots$. The proof is outlined in exercise 4.8.36.

- Although we have assumed that the order m is an integer, these relations hold for arbitrary $m \geq 0$.

Initial conditions revisited

Recall that the general solution to the vibrating circular membrane problem (1) - (3) is

$$\boxed{\begin{aligned} u(r,\theta,t) &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a_{mn}\cos m\theta + b_{mn}\sin m\theta\right)\cos c\lambda_{mn} t \\ &\quad + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a^*_{mn}\cos m\theta + b^*_{mn}\sin m\theta\right)\sin c\lambda_{mn} t. \end{aligned}}$$

and that the coefficients a_{mn} and b_{mn} are given by the Fourier-Bessel expansion of the initial shape $f(r,\theta)$:

$$f(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a_{mn}\cos m\theta + b_{mn}\sin m\theta\right).$$

As previously noted, the orthogonality relations allow us express the Fourier-Bessel coefficients as ratios of inner products.

Determining a_{mn} and b_{mn}

If $f(r,\theta)$ is "sufficiently nice," then we have

$$a_{mn} = \frac{\langle f, \phi_{mn}\rangle}{\langle \phi_{mn}, \phi_{mn}\rangle} = \frac{\displaystyle\int_0^{2\pi}\int_0^a f(r,\theta) J_m(\lambda_{mn} r)\cos(m\theta)\, r\, dr\, d\theta}{\displaystyle\int_0^{2\pi}\int_0^a J_m^2(\lambda_{mn} r)\, \cos^2(m\theta)\, r\, dr\, d\theta}$$

for $m \geq 0$, $n \geq 1$. Using the complementary orthogonality relation, the integral in the denominator is equal to

$$\int_0^{2\pi} \cos^2(m\theta)\, d\theta \int_0^a J_m^2(\lambda_{mn} r)\, r\, dr = \begin{cases} \pi a^2 J_1^2(\alpha_{0n}) & \text{if } m = 0, \\ \dfrac{\pi a^2}{2} J_{m+1}^2(\alpha_{mn}) & \text{if } m \geq 1. \end{cases}$$

We finally find that

$$a_{0n} = \frac{1}{\pi a^2 J_1^2(\alpha_{0n})} \int_0^{2\pi}\int_0^a f(r,\theta)\, J_0(\lambda_{0n} r)\, r\, dr\, d\theta,$$
$$a_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi}\int_0^a f(r,\theta)\, J_m(\lambda_{mn} r)\, \cos(m\theta)\, r\, dr\, d\theta,$$

and likewise

$$b_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi}\int_0^a f(r,\theta)\, J_m(\lambda_{mn} r)\, \sin(m\theta)\, r\, dr\, d\theta,$$

for $m, n = 1, 2, 3, \ldots$.

Formulas for a^*_{mn} and b^*_{mn}

Referring back to the Fourier-Bessel expansion of $g(x, y)$ and using the same line of reasoning leads to the analogous formulae

$$a^*_{0n} = \frac{1}{\pi c \alpha_{0n} a J_1^2(\alpha_{0n})} \int_0^{2\pi} \int_0^a g(r, \theta)\, J_0(\lambda_{0n} r)\, r\, dr\, d\theta,$$

$$a^*_{mn} = \frac{2}{\pi c \alpha_{mn} a J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \int_0^a g(r, \theta)\, J_m(\lambda_{mn} r)\, \cos(m\theta)\, r\, dr\, d\theta,$$

$$b^*_{mn} = \frac{2}{\pi c \alpha_{mn} a J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \int_0^a g(r, \theta)\, J_m(\lambda_{mn} r)\, \sin(m\theta)\, r\, dr\, d\theta,$$

for $m, n = 1, 2, 3, \ldots$.