Examples of the Circular Membrane Problem

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Recall:

In polar coordinates, the shape of a vibrating thin circular membrane of radius a can be modeled by

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a_{mn}\cos m\theta + b_{mn}\sin m\theta\right)\cos c\lambda_{mn} t + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a^*_{mn}\cos m\theta + b^*_{mn}\sin m\theta\right)\sin c\lambda_{mn} t$$

where

- J_m is the **Bessel function of order** m **of the first kind**,
- $\lambda_{mn} = \alpha_{mn}/a$, and
- α_{mn} is the nth positive zero of J_m.

The coefficients a_{mn}, b_{mn}, a^*_{mn} and b^*_{mn} are related to to the **initial shape** $f(r,\theta)$ and **initial velocity** $g(r,\theta)$ through integrals derived using orthogonality, e.g.

$$a_{mn} = \frac{2}{\pi a^2 J^2_{m+1}(\alpha_{mn})} \int_0^{2\pi} \int_0^a f(r,\theta)\, J_m(\lambda_{mn} r) \cos(m\theta)\, r\, dr\, d\theta,$$

for $m, n = 1, 2, 3, \ldots$.

- As much as possible, we would like to use standard integral calculus techniques to evaluate these integrals.
- This requires us to establish certain differentiation and integration identities involving Bessel functions.

Differentiation identities

Using the power series definition of $J_p(x)$, one can show that:

$$\frac{d}{dx}\left(x^p J_p(x)\right) = x^p J_{p-1}(x), \tag{1}$$

$$\frac{d}{dx}\left(x^{-p} J_p(x)\right) = -x^{-p} J_{p+1}(x). \tag{2}$$

The product rule and cancellation lead to

$$xJ_p'(x) + pJ_p(x) = xJ_{p-1}(x),$$
$$xJ_p'(x) - pJ_p(x) = -xJ_{p+1}(x).$$

Addition and subtraction of these identities then yield

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x),$$
$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x).$$

Integration identities

Integration of the differentiation identities (1) and (2) gives

$$\int x^{p+1} J_p(x)\,dx = x^{p+1} J_{p+1}(x) + C \tag{3}$$

$$\int x^{-p+1} J_p(x)\,dx = -x^{-p+1} J_{p-1}(x) + C.$$

- Exercises 4.2.12 and 4.3.9 give similar identities.
- Identities such as these can be used to evaluate certain integrals of the form

$$\int_0^a f(r) J_m(\lambda_{mn} r) r\,dr.$$

- Such integrals frequently occur in the coefficients of the solution to the vibrating membrane problem.

Example

Evaluate

$$\int x^{p+5} J_p(x)\, dx.$$

We integrate by parts, first taking

$$u = x^4 \qquad dv = x^{p+1} J_p(x)\, dx$$
$$du = 4x^3\, dx \qquad v = x^{p+1} J_{p+1}(x),$$

which gives

$$\int x^{p+5} J_p(x)\, dx = x^{p+5} J_{p+1}(x) - 4 \int x^{p+4} J_{p+1}(x)\, dx.$$

Now integrate by parts again with

$$u = x^2 \qquad dv = x^{p+2} J_{p+1}(x)\,dx$$
$$du = 2x\,dx \qquad v = x^{p+2} J_{p+2}(x),$$

to get

$$\begin{aligned}\int x^{p+5} J_p(x)\,dx &= x^{p+5} J_{p+1}(x) - 4\int x^{p+4} J_{p+1}(x)\,dx \\ &= x^{p+5} J_{p+1}(x) - 4\left(x^{p+4} J_{p+2}(x) - 2\int x^{p+3} J_{p+2}(x)\,dx\right) \\ &= \boxed{x^{p+5} J_{p+1}(x) - 4x^{p+4} J_{p+2}(x) + 8x^{p+3} J_{p+3}(x) + C.}\end{aligned}$$

A radially symmetric example

Example

Solve the vibrating membrane problem with $a = c = 1$ and initial conditions

$$f(r,\theta) = 1 - r^4, \qquad g(r,\theta) = 0.$$

Because $g(r,\theta) = 0$, we immediately find that $a^*_{mn} = b^*_{mn} = 0$ for all m and n. Moreover, because $f(r,\theta) = f(r)$ does not depend on θ (f is **radially symmetric**)

$$\begin{aligned} a_{mn} &= \frac{2}{\pi J^2_{m+1}(\alpha_{mn})} \int_0^{2\pi}\int_0^1 f(r) J_m(\alpha_{mn} r)\cos(m\theta) r\, dr d\theta \\ &= \frac{2}{\pi J^2_{m+1}(\alpha_{mn})} \underbrace{\int_0^{2\pi} \cos(m\theta)\, d\theta}_{0} \int_0^a f(r) J_m(\alpha_{mn} r) r\, dr \\ &= 0 \qquad \text{if } m \geq 1. \end{aligned}$$

Likewise, $b_{mn} = 0$ for all $m, n \geq 1$. Finally, we have

$$
\begin{aligned}
a_{0n} &= \frac{1}{\pi J_1^2(\alpha_{0n})} \int_0^{2\pi} \int_0^1 f(r) J_0(\alpha_{0n} r) r \, dr d\theta \\
&= \frac{2}{J_1^2(\alpha_{0n})} \underbrace{\int_0^1 (1 - r^4) J_0(\alpha_{0n} r) r \, dr}_{\text{substitute} x = \alpha_{0n} r} \\
&= \frac{2}{\alpha_{0n}^2 J_1^2(\alpha_{0n})} \int_0^{\alpha_{0n}} \left(1 - \frac{x^4}{\alpha_{0n}^4}\right) J_0(x) x \, dx \\
&= \frac{2}{\alpha_{0n}^2 J_1^2(\alpha_{0n})} \left(\underbrace{\int_0^{\alpha_{0n}} x J_0(x) \, dx}_{A} - \frac{1}{\alpha_{0n}^4} \underbrace{\int_0^{\alpha_{0n}} x^5 J_0(x) \, dx}_{B} \right).
\end{aligned}
$$

The integral A is

$$\int_0^{\alpha_{0n}} xJ_0(x)\,dx = xJ_1(x)\Big|_0^{\alpha_{0n}} = \alpha_{0n}J_1(\alpha_{0n}).$$

The integral B is

$$\begin{aligned}\int_0^{\alpha_{0n}} x^5 J_0(x)\,dx &= x^5J_1(x) - 4x^4J_2(x) + 8x^3J_3(x)\Big|_0^{\alpha_{0n}}\\ &= \alpha_{0n}^5J_1(\alpha_{0n}) - 4\alpha_{0n}^4J_2(\alpha_{0n}) + 8\alpha_{0n}^3J_3(\alpha_{0n}).\end{aligned}$$

Consequently

$$A - \frac{1}{\alpha_{0n}^4}B = 4J_2(\alpha_{0n}) - 8\alpha_{0n}^{-1}J_3(\alpha_{0n}).$$

Putting these all together we get

$$\begin{aligned} a_{0n} &= \frac{2}{\alpha_{0n}^2 J_1^2(\alpha_{0n})}\left(4J_2(\alpha_{0n}) - 8\alpha_{0n}^{-1}J_3(\alpha_{0n})\right) \\ &= \frac{8\left(\alpha_{0n}J_2(\alpha_{0n}) - 2J_3(\alpha_{0n})\right)}{\alpha_{0n}^3 J_1^2(\alpha_{0n})}. \end{aligned}$$

Because $a_{mn} = b_{mn} = 0$ for $m \geq 1$, we find that

$$\boxed{u(r,\theta,t) = \sum_{n=1}^{\infty} \frac{8\left(\alpha_{0n}J_2(\alpha_{0n}) - 2J_3(\alpha_{0n})\right)}{\alpha_{0n}^3 J_1^2(\alpha_{0n})} J_0(\alpha_{0n}r)\cos(\alpha_{0n}t).}$$

- The coefficients a_{0n} can actually be expressed in terms of J_1 only, by using the final differentiation identity from above.
- Any time the initial conditions are radially symmetric, the solution takes a similar form.

A non-symmetric example

Example

Solve the vibrating membrane problem with $a = c = 1$ and initial conditions

$$f(r,\theta) = r(1-r^4)\cos\theta, \qquad g(r,\theta) = 0.$$

As before, we have $a^*_{mn} = b^*_{mn} = 0$ for all m, n. We also have

$$\begin{aligned} b_{mn} &= \frac{2}{\pi J^2_{m+1}(\alpha_{mn})} \int_0^{2\pi}\int_0^1 \underbrace{r(1-r^4)\cos\theta}_{f(r,\theta)} J_m(\alpha_{mn} r)\sin(m\theta) r\, dr d\theta \\ &= \frac{2}{\pi J^2_{m+1}(\alpha_{mn})} \underbrace{\int_0^{2\pi} \cos\theta \sin(m\theta)\, d\theta}_{0} \int_0^1 r(1-r^4) J_m(\alpha_{mn} r) r\, dr \\ &= 0 \qquad \text{for all } m, n. \end{aligned}$$

Additionally,

$$\begin{aligned} a_{0n} &= \frac{1}{\pi J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi}\int_0^1 r(1-r^4)\,\cos\theta J_m(\alpha_{mn} r) r\,dr d\theta \\ &= \frac{1}{\pi J_{m+1}^2(\alpha_{mn})} \underbrace{\int_0^{2\pi} \cos\theta\, d\theta}_{0} \int_0^1 r(1-r^4) J_m(\alpha_{mn} r) r\,dr \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} a_{mn} &= \frac{2}{\pi J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi}\int_0^1 r(1-r^4)\,\cos\theta J_m(\alpha_{mn} r)\cos(m\theta) r\,dr d\theta \\ &= \frac{2}{\pi J_{m+1}^2(\alpha_{mn})} \underbrace{\int_0^{2\pi} \cos\theta\,\cos(m\theta)\, d\theta}_{A} \int_0^1 r(1-r^4) J_m(\alpha_{mn} r) r\,dr. \end{aligned}$$

The integral A is zero unless $m = 1$, in which case it's equal to π. In this case

$$\begin{aligned} a_{1n} &= \frac{2}{J_2^2(\alpha_{1n})} \int_0^1 r(1 - r^4) J_1(\alpha_{1n} r) r \, dr \\ &= \frac{2}{J_2^2(\alpha_{1n})} \left(\int_0^1 r^2 J_1(\alpha_{1n} r) \, dr - \int_0^1 r^6 J_1(\alpha_{1n} r) \, dr \right). \end{aligned}$$

Substituting $x = \alpha_{1n} r$ and proceeding as before one can show

$$\begin{aligned} \int_0^1 r^2 J_1(\alpha_{1n} r) \, dr &= \frac{J_2(\alpha_{1n})}{\alpha_{1n}} \\ \int_0^1 r^6 J_1(\alpha_{1n} r) \, dr &= \frac{J_2(\alpha_{1n})}{\alpha_{1n}} - \frac{4 J_3(\alpha_{1n})}{\alpha_{1n}^2} + \frac{8 J_4(\alpha_{1n})}{\alpha_{1n}^3}. \end{aligned}$$

Assembling these formulae gives

$$\begin{aligned} a_{1n} &= \frac{2}{J_2^2(\alpha_{1n})}\left(\frac{4J_3(\alpha_{1n})}{\alpha_{1n}^2} - \frac{8J_4(\alpha_{1n})}{\alpha_{1n}^3}\right) \\ &= \frac{8\left(\alpha_{1n}J_3(\alpha_{1n}) - 2J_4(\alpha_{1n})\right)}{\alpha_{1n}^3 J_2^2(\alpha_{1n})}. \end{aligned}$$

Since all the other coefficients are zero,

$$\boxed{u(r,\theta,t) = \cos\theta \sum_{n=1}^{\infty} \frac{8\left(\alpha_{1n}J_3(\alpha_{1n}) - 2J_4(\alpha_{1n})\right)}{\alpha_{1n}^3 J_2^2(\alpha_{1n})} J_1(\alpha_{1n}r)\cos(\alpha_{1n}t).}$$

As in the previous example, one may express the coefficients entirely in terms of J_2 using the final differentiation identity.

Remarks

- In general, one should **not** expect the solution to reduce to a single series.
- If $g(r,\theta) \neq 0$:
  - Computations similar to those above must be carried out to determine a^*_{mn} and b^*_{mn}.
  - This adds an additional series to the solution.
- If $f(r,\theta)$ or $g(r,\theta)$ is "too complicated," one can use Maple to help evaluate the integrals defining the coefficients of the solution.

A "complicated" example

Example

Solve the vibrating membrane problem with $a = 2$, $c = 1$ and initial conditions

$$f(r,\theta) = 0, \qquad g(r,\theta) = r^2(2-r)\sin^8\left(\frac{\theta}{2}\right).$$

Since $f \equiv 0$, $a_{mn} = 0$, $b_{mn} = 0$. We also have $b^*_{mn} = 0$ since

$$b^*_{mn} = \frac{2}{\pi\alpha_{mn}2J^2_{m+1}(\alpha_{mn})}\underbrace{\int_0^{2\pi}\sin^8\left(\frac{\theta}{2}\right)\sin(m\theta)\,d\theta}_{0}\int_0^2 r^2(2-r)J_m(\lambda_{mn}r)r\,dr,$$

because the θ integrand is odd and 2π-periodic.

The other coefficients are given by

$$a_{0n}^* = \frac{1}{\pi\alpha_{0n} 2J_1^2(\alpha_{0n})} \int_0^{2\pi} \sin^8\left(\frac{\theta}{2}\right)\, d\theta \int_0^2 r^2(2-r)J_0(\lambda_{0n}r)r\,dr,$$

$$a_{mn}^* = \frac{2}{\pi\alpha_{mn} 2J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \sin^8\left(\frac{\theta}{2}\right) \cos(m\theta)\, d\theta \int_0^2 r^2(2-r)J_m(\lambda_{mn}r)r\,dr.$$

These integrals and the overall solution are best left to computer evaluation.