Exercise 1. Determine the order of each of the following PDEs and state whether or not they are linear. If an equation is linear, state whether or not it is homogeneous.

c. $u_{xx} + (x^2 + y)u_{yy} = 0$

e. $u_x u_y + u_z = -u_{xyz}$

g. $(u_{xx} + u_x)_{tt} + e^{x+y} u_y = 0$

h. $\dfrac{\partial^2 u}{\partial x_1 \partial x_2} + \dfrac{\partial^2 u}{\partial x_2 \partial x_3} + \dfrac{\partial^2 u}{\partial x_1 \partial x_3} = u + x_1 x_2 x_3$

Exercise 2. This problem concerns the partial differential equation

$$u_{xx} + 3u_{xy} + 2u_{yy} = 0. \tag{1}$$

a. If F and G are twice differentiable functions, show that

$$u(x, y) = F(2x - y) + G(x - y) \tag{2}$$

is a solution to (1).

b. Use a linear change of variables to show that every solution to (1) has the form (2).

c. Find the solution to (1) that satisfies the initial conditions

$$u(x, 0) = \frac{x}{x^2 + 1} \text{ and } u_y(x, 0) = 0 \text{ for all } x.$$

Exercise 3. Show that the general solution to $u_{xy} + u_x = 0$ has the form $u(x, y) = F(y) + e^{-y} G(x)$. [*Suggestion:* Notice that $u_{xy} + u_x = (u_y + u)_x$.]

Exercise 4. Find the general solution to the partial differential equation

$$\frac{\partial u}{\partial x} + \left(\frac{4y}{x} - x^2 + 1\right)\frac{\partial u}{\partial y} = 0.$$

Exercise 5. Find the solution to the wave equation $u_{tt} = 9u_{xx}$ that satisfies the boundary and initial conditions

$$u(0,t) = u(2,t) = 0 \text{ for all } t,$$
$$u(x,0) = \sin\left(\frac{\pi x}{2}\right) \text{ for all } 0 \leq x \leq 2,$$
$$u_t(x,0) = x \text{ for all } 0 \leq x \leq 2.$$

Exercise 6. Let

$$\begin{aligned} f_1(x) &= 1, \\ f_2(x) &= 2x-1, \\ f_3(x) &= 6x^2-6x+1, \\ f_4(x) &= 20x^3-30x^2+12x-1. \end{aligned}$$

a. Verify that the polynomials $f_1(x) = 1$, $f_2(x) = 2x-1$, $f_3(x) = 6x^2-6x+1$ and $f_4(x) = 20x^3-30x^2+12x-1$ are pairwise orthogonal on $[0,1]$.

b. Let $p(x) = x^3 - 2$. Use part **a** to write p as a linear combination of f_1, f_2, f_3 and f_4. [*Suggestion:* Recall that if

$$p = a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4$$

then $a_j = \langle p, f_j\rangle / \langle f_j, f_j\rangle$.]

c. Explain why the procedure of part **b** *fails* if we take $p(x) = x^5 - 2x + 1$.

Exercise 7. Let

$$f(x) = \begin{cases} [x] & \text{if } -1 \leq x < 2, \\ f(x+3) & \text{otherwise.} \end{cases}$$

a. What function does the Fourier series of $f(x)$ converge to?

b. Find the Fourier series for $f(x)$:

i. By computing the Fourier coefficients using Euler's formulas.

ii. By realizing $f(x)$ as a linear combination of translations, dilations or reflections of functions with known Fourier series.