Partial Differential Equations — Exam 2 Review Exercises

Spring 2012

Exercise 1. Let

$$f(x) = \begin{cases} x+1 & \text{if } 0 \le x < 1, \\ 2 & \text{if } 1 < x \le 2. \end{cases}$$

a. Find the sine series expansion of $f(x)$.

b. Find the cosine series expansion of $f(x)$.

c. Sketch the graphs of the functions to which these series converge, on the interval $-4 \le x \le 4$.

Exercise 2. Find the complex form of the Fourier series for the $2p$-periodic function

$$g(x) = \begin{cases} -\frac{2}{p}(x + p/2) & \text{if } -p \le x < 0, \\ -\frac{2}{p}(x - p/2) & \text{if } 0 \le x < p, \\ g(x+2p) & \text{otherwise} \end{cases}$$

by:

a. Using the integral formulas for the coefficients.

b. Using the real form of the Fourier series given in textbook exercise 2.3.7.

Exercise 3. Solve the boundary value problem

$$\begin{aligned} u_{tt} &= 16u_{xx}, && 0 < x < 3,\ t > 0 \\ u(0,t) &= u(3,t) = 0, && t \ge 0, \\ u(x,0) &= \begin{cases} x & \text{if } 0 \le x \le 2, \\ 2(3-x) & \text{if } 2 < x \le 3, \end{cases} \\ u_t(x,0) &= \sin \pi x + \frac{1}{2}\sin 2\pi x + 3 \sin 3\pi x, && 0 \le x \le 3. \end{aligned}$$

Exercise 4. A bar of length π and thermal diffusivity 1 is heated so that its temperature at a point x is given by $h(x) = 50x(\pi - x)$. The lateral surface of the bar is then insulated, the left end is held at a temperature of 100, and the right end is held at a temperature of 50. Determine the temperature in the bar at any later time.

Exercise 5. Use separation of variables to reduce the PDE boundary value problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= g\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right), & & 0 < x < L,\ t > 0 \\ u(L,t) &= 0, & & t \geq 0 \end{aligned}$$

to a system of two ODEs with boundary conditions. Here g and L are unspecified positive constants.

Exercise 6. Consider the heat boundary value problem given by

$$\begin{aligned} u_t &= c^2 u_{xx}, & & 0 < x < L,\ t > 0 \\ u_x(0,t) &= \kappa_1 u(0,t), & & t \geq 0, \\ u_x(L,t) &= -\kappa_2 u(L,t), & & t \geq 0, \\ u(x,0) &= f(x), & & 0 \leq x \leq L, \end{aligned}$$

in which κ_1 and κ_2 are positive constants.

a. Provide a physical interpretation of this problem.

b. Use separation of variables to obtain a pair of ODE boundary value problems.

c. Show that the separation constant k in part **b** must be negative: $k = -\mu^2$. Find the equation satisfied by μ, and show that this equation has infinitely many positive solutions

$$\mu_1 < \mu_2 < \mu_3 < \cdots .$$

d. Given an n, find the normal mode corresponding to μ_n.

Exercise 7. Let

$$\begin{aligned} X_0(x) &= e^{-x}, & & \\ X_n(x) &= n\pi \cos n\pi x - \sin n\pi x, & & n \geq 1. \end{aligned}$$

Show that the functions $X_0, X_1, X_2, \ldots$ are pairwise orthogonal on the interval $[0,1]$.

Exercise 8. A thin elastic membrane is stretched and fixed to a square frame with unit side length. The membrane is deformed to have shape given by $f(x,y) = \sin 3\pi x \sin \pi y$ and is pulled up everywhere with unit velocity. Find a function that describes the resulting motion of the membrane.