Partial Differential Equations — Exam 3 Review Exercises
Spring 2012

Exercise 1. Solve the Dirichlet problem $\nabla^2 u = 0$ on the rectangle $[0,2]\times[0,2]$ subject to the boundary conditions

$$\begin{aligned} u(0,y) &= 50y^2, & u(2,y) &= 50(2-y), & 0 \le y \le 2,\\ u(x,0) &= 50x, & u(x,2) &= 50(2-x)^2, & 0 \le x \le 2. \end{aligned}$$

Exercise 2. Show that the substitution $z = 2e^{-x/2}$ transforms the ODE

$$y'' + e^{-x}y = 0$$

into a Bessel equation.

Exercise 3.

a. Use the result of Example 1 in section 4.7 and equation (6) to show that

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right).$$

b. Use equation (6) from section 4.8 to express J_5 in terms of J_0 and J_1.

Exercise 4. An elastic membrane is stretched across a circular frame of radius 5 (centered at the origin in the xy-plane) and is imparted with an initial velocity described in polar coordinates by

$$g(r,\theta) = (25 - r^2)r^3\cos 3\theta.$$

Give an expression for the shape of the membrane at all later times.

Exercise 5. [5% extra credit] A perfectly elastic membrane is attached to a rigid circular frame of radius 1. Its center is pulled to a height of 0.1 and then released. Assuming that the constant c in the wave equation is equal to 1, use Maple to produce a (smooth) animation of the resulting motion of the membrane for the next 10 units of time.

Exercise 6. Consider the boundary value problem

$$(1 - x^2)y'' - 3xy' + n(n+2)y = 0, \qquad -1 < x < 1, \tag{1}$$
$$y \text{ bounded for } -1 < x < 1. \tag{2}$$

a. Show that (1)–(2) is a singular Sturm-Liouville problem. Be sure to give **every** reason that this problem is not regular.

b. Prove orthogonality of the eigenfunctions of (1)–(2).

Exercise 7. Consider the boundary value problem

$$\begin{aligned} &u_{xx} - xu_t = 0, && t > 0, \quad 0 < x < 1, \\ &u(0,t) = u(1,t) = 0, && \\ &u(x,0) = f(x), && 0 \le x \le 1. \end{aligned}$$

- Show that separation of variables yields a regular Sturm-Liouville problem.
- Use S-L theory to give the general solution to this problem (you may assume that f is a C^1 function).