Exercise 1. We utilize the general solution to the Dirichlet problem in rectangle given in the textbook on page 168. In the notation used there and on page 167, we have $a = b = 2$, $f_1(x) = 50x$, $f_2(x) = 50(2-x)^2$, $g_1(y) = 50y^2$ and $g_2(y) = 50(2-y)$. It follows that the coefficients in the series solution are

$$\begin{aligned}
A_n &= \frac{1}{\sinh n\pi}\int_0^2 50x \sin\frac{n\pi x}{2}\,dx = \frac{200(-1)^{n+1}}{n\pi\sinh n\pi},\\
B_n &= \frac{1}{\sinh n\pi}\int_0^2 50(2-x)^2 \sin\frac{n\pi x}{2}\,dx = \frac{400\left(2((-1)^n-1)+n^2\pi^2\right)}{n^3\pi^3\sinh n\pi},\\
C_n &= \frac{1}{\sinh n\pi}\int_0^2 50y^2 \sin\frac{n\pi y}{2}\,dy = \frac{400\left(2((-1)^n-1)+(-1)^{n+1}n^2\pi^2\right)}{n^3\pi^3\sinh n\pi},\\
D_n &= \frac{1}{\sinh n\pi}\int_0^2 50(2-y) \sin\frac{n\pi y}{2}\,dy = \frac{200}{n\pi\sinh n\pi}.
\end{aligned}$$

The complete solution is then given by

$$\begin{aligned}
u(x,y) = &\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{2}\sinh\frac{n\pi(2-y)}{2} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2}\sinh\frac{n\pi y}{2}\\
&+ \sum_{n=1}^{\infty} C_n \sinh\frac{n\pi(2-x)}{2}\sin\frac{n\pi y}{2} + \sum_{n=1}^{\infty} D_n \sinh\frac{n\pi x}{2}\sin\frac{n\pi y}{2},
\end{aligned}$$

with A_n, B_n, C_n and D_n as above.

Exercise 2. This is simply an exercise in using the chain rule. If $z = 2e^{-x/2}$, then on applying the chain rule we have

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = -\frac{dy}{dz}e^{-x/2}.$$

Another application of the chain rule (with the product rule) then yields

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(-\frac{dy}{dz}e^{-x/2}\right) = -\frac{d}{dx}\left(\frac{dy}{dz}\right)e^{-x/2} + \frac{1}{2}\frac{dy}{dz}e^{-x/2}\\
&= -\frac{d^2y}{dz^2}\frac{dz}{dx}e^{-x/2} + \frac{1}{2}\frac{dy}{dz}e^{-x/2} = \frac{d^2y}{dz^2}e^{-x} + \frac{1}{2}\frac{dy}{dz}e^{-x/2}.
\end{aligned}$$

Since $e^{-x/2} = z/2$, this implies that

$$\frac{d^2y}{dx^2} = \frac{z^2}{4}\frac{d^2y}{dz^2} + \frac{z}{4}\frac{dy}{dz}.$$

Substituting this into the differential equation $y'' + e^{-x}y = 0$ and again using the fact that $e^{-x/2} = z/2$, we get

$$\frac{z^2}{4}\frac{d^2y}{dz^2} + \frac{z}{4}\frac{dy}{dz} + \frac{z^2}{4}y = 0.$$

Multiplying through by 4 finally gives us

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + z^2y = 0,$$

which is Bessel's equation of order 0.

Exercise 3. The central idea in both parts of this exercise is to use equation (6) of section 4.8, namely

$$J_{p+1} = \frac{2p}{x}J_p(x) - J_{p-1}(x), \tag{1}$$

to express the given Bessel function in terms of those of lower order.

a. Equation (1) with $p = 3/2$ tells us that

$$J_{5/2}(x) = \frac{3}{x}J_{3/2}(x) - J_{1/2}(x). \tag{2}$$

Our goal is to get formulas for $J_{1/2}(x)$ and $J_{3/2}(x)$ and plug them in to (2). According to Example 1 in section 4.7,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x. \tag{3}$$

We now need a similar formula for $J_{3/2}(x)$. For this we turn to identity (4) of section 4.8. If we divide by $-x$ and set $p = 1/2$, we find that

$$\begin{aligned} J_{3/2}(x) &= -J'_{1/2}(x) + \frac{1}{2x}J_{1/2}(x) \\ &= -\sqrt{\frac{2}{\pi x}}\cos x + \frac{1}{2x}\sqrt{\frac{2}{\pi x}}\sin x + \frac{1}{2x}\sqrt{\frac{2}{\pi x}}\sin x \\ &= -\sqrt{\frac{2}{\pi x}}\cos x + \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x, \end{aligned}$$

where we have used the expression (3) for $J_{1/2}(x)$. If we plug this and (3) into (2) we obtain

$$\begin{aligned} J_{5/2}(x) &= -\frac{3}{x}\sqrt{\frac{2}{\pi x}}\cos x + \frac{3}{x^2}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\sin x \\ &= \sqrt{\frac{2}{\pi x}}\left(\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right), \end{aligned}$$

which is what we were asked to show.

b. This is simply a repeated use of (1). Starting with $p = 4$ we get

$$J_5(x) = \frac{8}{x}J_4(x) - J_3(x). \tag{4}$$

Taking $p = 3$ gives

$$J_4(x) = \frac{6}{x} J_3(x) - J_2(x),$$

and plugging this into (4) and simplifying we find that

$$J_5(x) = \left(\frac{48}{x^2} - 1 \right) J_3(x) - \frac{8}{x} J_2(x).$$

If we repeat this process using $p = 2$ and then $p = 1$ in (1), we eventually find that

$$J_5(x) = \left(1 - \frac{72}{x^2} + \frac{384}{x^4} \right) J_1(x) + \left(\frac{12}{x} - \frac{192}{x^3} \right) J_0(x).$$

[**Remark:** There is actually a nice way to make the procedure just described much more efficient (i.e. amenable to hand computation). The idea is to write the relationship (1) in matrix form as

$$\begin{pmatrix} J_{p+1}(x) \\ J_p(x) \end{pmatrix} = \begin{pmatrix} 2p/x & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} J_p(x) \\ J_{p-1}(x) \end{pmatrix}.$$

If we let

$$A_p = \begin{pmatrix} 2p/x & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{v}_p = \begin{pmatrix} J_p(x) \\ J_{p-1}(x) \end{pmatrix},$$

then this becomes

$$\mathbf{v}_{p+1} = A_p \mathbf{v}_p.$$

It then follows that

$$\mathbf{v}_5 = A_4 \mathbf{v}_4 = A_4 A_3 \mathbf{v}_3 = A_4 A_3 A_2 \mathbf{v}_2 = A_4 A_3 A_2 A_1 \mathbf{v}_1.$$

That is,

$$\begin{pmatrix} J_5(x) \\ J_4(x) \end{pmatrix} = A_4 A_3 A_2 A_1 \begin{pmatrix} J_1(x) \\ J_0(x) \end{pmatrix}.$$

Computing the matrix product $A_4 A_3 A_2 A_1$ (a relatively straightforward procedure), this immediately expresses $J_5(x)$ (and $J_4(x)$, too) in terms of $J_0(x)$ and $J_1(x)$. This same idea can be used to quickly express any given Bessel function of the first kind in terms of Bessel functions of lower order.]

Exercise 4. As with Exercise 1, we simply refer to the general solution to the vibrating membrane problem as given on pages 211 and 213 in our textbook. The statement of the problem implies that the initial shape of the membrane is $f(r, \theta) = 0$ (i.e. the membrane is initially flat), which according to the integral formulae on page 211 immediately tells us that

$$a_{mn} = b_{mn} = 0$$

for all m and n. Moreover, using the given radius and initial velocity, the integral factor appearing in the formula for a_{0n}^* is

$$\int_0^5 \int_0^{2\pi} (25 - r^2) r^3 \cos 3\theta J_0(\lambda_{0n} r) r \, d\theta \, dr = \int_0^5 *** \, dr \underbrace{\int_0^{2\pi} \cos 3\theta \, d\theta}_{0} = 0$$

(we have simply written the r-integrand as $***$ since it is clearly immaterial to this computation). This implies that $a^*_{0n} = 0$ for all n. Likewise, the integral factor appearing in b^*_{mn} separates as

$$\int_0^5 *** \, dr \underbrace{\int_0^{2\pi} \cos 3\theta \, \sin m\theta \, d\theta}_{0} = 0,$$

which implies that $b^*_{mn} = 0$ for all m and n (the fact that the θ integral vanishes can be observed in one of several ways: by direct antidifferentiation; by observing that the integrand is an odd 2π-periodic function, and we are integrating over a complete period; or by utilizing previously established orthogonality results).

We now turn to the computation of the a^*_{mn} coefficients. Again focusing only on the integral factor, we find that it separates to become

$$\underbrace{\int_0^5 (25 - r^2) r^4 J_m(\lambda_{mn} r) \, dr}_{A} \underbrace{\int_0^{2\pi} \cos 3\theta \, \cos m\theta \, d\theta}_{B}.$$

The integral B is equal to 0 unless $m = 3$ (as above, there are a few ways to verify this, the simplest probably being orthogonality), in which case it is equal to π. This means $a^*_{mn} = 0$, unless $m = 3$. Setting $m = 3$, we can quickly evaluate the integral A by taking advantage of the "useful identity" on page 211 of the textbook. It gives

$$\begin{aligned} \int_0^5 (25 - r^2) r^4 J_3(\lambda_{3n} r) \, dr &= \int_0^5 (25 - r^2) r^4 J_3 \left(\lambda_{3n} r\right) \, dr \\ &= \frac{2 \cdot 5^7}{\alpha_{3n}^2} J_5(\alpha_{3n}). \end{aligned}$$

Putting these computations together we find that (taking a quick glance at equation (18) on page 213)

$$\begin{aligned} a^*_{3n} &= \frac{2}{5\pi c \alpha_{3n} J_4^2(\alpha_{3n})} AB \\ &= \frac{4 \cdot 5^6 J_5(\alpha_{3n})}{c \alpha_{3n}^3 J_4^2(\alpha_{3n})}. \end{aligned}$$

This can be further simplified (although this isn't strictly necessary) using identity (1) with $p = 4$, which tells us that

$$J_5(\alpha_{3n}) = \frac{8}{\alpha_{3n}} J_4(\alpha_{3n}) - J_3(\alpha_{3n}) = \frac{8 J_4(\alpha_{3n})}{\alpha_{3n}},$$

since $J_3(\alpha_{3n}) = 0$. Using this in our previous expression and simplifying slightly, we end up with

$$a^*_{3n} = \frac{500000}{c \alpha_{3n}^4 J_4(\alpha_{3n})}.$$

Finally, we conclude that the shape of the membrane at time t is given by

$$\begin{aligned} u(r,\theta,t) &= \sum_{n=1}^{\infty} a_{3n}^* J_3(\lambda_{3n} r)\, \cos 3\theta\, \sin c\lambda_{3n} t \\ &= \frac{500000 \cos 3\theta}{c} \sum_{n=1}^{\infty} \frac{1}{\alpha_{3n}^4 J_4(\alpha_{3n})} J_3\left(\frac{\alpha_{3n}}{5} r\right) \sin\left(\frac{c\alpha_{3n} t}{5}\right) \end{aligned}$$

(the "elasticity constant," c, was unspecified, so it must appear in our solution as a parameter).

Exercise 5. This one's extra credit. No hints!

Exercise 6.

a. The given differential equation is not in S-L form. However, if we multiply it by $(1-x^2)^{1/2}$ it becomes

$$(1-x^2)^{3/2} y' - 3x(1-x^2)^{1/2} y' + n(n+2)(1-x^2)^{1/2} y = 0,$$

which is the same as

$$((1-x^2)^{3/2} y')' + n(n+2)(1-x^2)^{1/2} y = 0.$$

This *is* in S-L form, with $p(x) = (1-x^2)^{3/2}$, $q(x) = 0$ and $r(x) = (1-x^2)^{1/2}$ (with parameter $\lambda = n(n+2)$). The given problem is singular because:

- The boundary conditions are not of the correct form (see equation (2) of section 6.2).
- The function $p(x)$ is not positive for $-1 \le x \le 1$, since $p(-1) = p(1) = 0$.
- The function $r(x)$ is not positive for $-1 \le x \le 1$, since $r(-1) = r(1) = 0$.

b. Suppose that y_1 and y_2 are eigenfunctions of the problem in question, with distinct eigenvalues. In class we showed that y_1 and y_2 are guaranteed to be orthogonal (on the interval $[-1,1]$, with respect to the weight function $r(x) = (1-x^2)^{1/2}$) provided that

$$p(x)\left(y_1'(x) y_2(x) - y_2'(x) y_1(x)\right)\Big|_{-1}^{1} = 0. \tag{5}$$

Because $p(-1) = p(1) = 0$ (as noted above), this is immediate, thereby verifying orthogonality.

Exercise 7. Separating variables, we assume that $u(x,t) = X(x)T(t)$ and plug into the given PDE, yielding

$$X''T - xXT' = 0.$$

Dividing by xXT, this becomes

$$\frac{X''}{xX} - \frac{T'}{T} = 0,$$

or

$$\frac{X''}{xX} = \frac{T'}{T} = -\lambda$$

for some constant λ, since the first two terms depend on different independent variables. Reorganizing slightly, we obtain the two ODEs

$$X'' + \lambda x X = 0, \tag{6}$$

$$T' + \lambda T = 0. \tag{7}$$

Because we seek nontrivial solutions, the given boundary conditions also tell us that

$$X(0) = X(1) = 0.$$

The equation for X is in S-L form with $p(x) = 1$, $q(x) = 0$ and $r(x) = x$.

The conditions $X(0) = 0$ and $X(1) = 0$ are "regular" (see equation (2) of section 6.2), but the fact that $r(0) = 0$ means that the boundary value problem for X is actually singular.[1] Nonetheless, the fact that the boundary conditions are regular implies the eigenfunctions for distinct eigenvalues are orthogonal on $[0, 1]$ with respect to the weight $r(x) = x$. Moreover, one can show directly that the eigenvalues form an increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

that tends to infinity, that (up to a scalar multiple) the eigenfunction $X_n(x)$ associated with λ_n is unique, and that the "Fourier" convergence theorem holds for these eigenfunctions.[2] Taking this as given, equation (7) tells us that

$$T(t) = T_n(t) = e^{-\lambda_n t}$$

so that the normal modes of the original boundary value problem are

$$u_n(x,t) = X_n(x)e^{-\lambda_n t},$$

and the general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) e^{-\lambda_n t}.$$

The initial condition gives

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n X_n(x),$$

with (according to the "Fourier" convergence result quoted above)

$$c_n = \frac{\displaystyle\int_0^1 f(x)X_n(x)x\,dx}{\displaystyle\int_0^1 X_n^2(x)x\,dx}.$$

[1] According to my notes, I originally intended to use the interval $1 \le x \le 2$. This would have kept the problem regular, since then p and r would have been positive throughout the interval.

[2] This is *automatically* true of any *regular* S-L problem, and a brief sketch of the derivation of these facts in this particular (singular) case is given below.

[**Remark:** Changing variables (specifically by first substituting $y = x^{-1/2}X$, followed by setting $s = 2x^{3/2}/3$), one can transform the equation $X'' + \lambda x X = 0$ into the parametric Bessel equation of order 1/3. Imposing the boundary conditions one can then show that nonzero solutions are possible only for $\lambda > 0$, and after back substitution one can show that in this case the eigenvalues and eigenfunctions are

$$\lambda_n = \frac{9\alpha_{1/3,n}^2}{4},$$
$$X_n(x) = x^{1/2} J_{1/3}\left(\alpha_{1/3,n} x^{3/2}\right)$$

for $n = 1, 2, 3, \ldots$. Aside from the completeness statement (i.e. that "Fourier" convergence holds), this proves the claims made above.]