Math 3357 Spring 2012

Partial Differential Equations

First Midterm Exam

Tuesday, February 7

Your name (please print):

Instructions: This is a closed book, closed notes exam. You may use a calculator, **but the use of other electronic devices such as cell phones, mp3 players, etc. is not permitted.** Unless indicated otherwise, you must justify all of your answers to receive credit. Unjustified answers and/or disorganized or otherwise illegible work will receive partial credit at best. Notation is important, and points will be deducted for incorrect use. Please do all of your work on the paper provided.

The Honor Code requires that you neither give nor receive any aid on this exam.

Please indicate that you have read and understood these guidelines by signing your name in the space provided:

Pledged: ______________________________

Do not write below this line

Problem	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Points	11	5	12	15	14	6	10	8	14	5
Score										

Total:____________________

1. Determine the order of each of the following PDEs and state whether or not they are linear. If an equation is linear, state whether or not it is homogeneous.

a. $(p(x)u_x)_x - r(x)u_{tt} = 0$

b. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^4 u}{\partial x^2 \partial t^2} + \dfrac{\partial u}{\partial t} = \sin(t)$

c. $\alpha^2 \left(u_r r + \dfrac{1}{r} u_r + \dfrac{1}{r^2} u_{\theta\theta} \right) = u_t$

d. $\left(\dfrac{\partial u}{\partial y} \right)^2 = \dfrac{\partial^3 u}{\partial x^3}$

2. Verify that $u(x,t) = \dfrac{x}{t+1}$ is a solution to $u_t + uu_x = 0$.

3. Find the general solution to the partial differential equation

$$y\frac{\partial u}{\partial x} = x\frac{\partial u}{\partial y}.$$

4. Consider the partial differential equation

$$u_x - 2u_y = u^2. \tag{1}$$

a. Use a linear change of variables to reduce (1) to the equation

$$u_\alpha = u^2. \tag{2}$$

b. Solve equation (2).

c. Use parts **a** and **b** to solve (1).

5. Find the solution of the initial value problem

$$
\begin{aligned}
&u_{tt} - 25u_{xx} = 0,\\
&u(x,0) = e^{-x^2} \text{ for all } x,\\
&u_t(x,0) = \cos\left(\frac{x}{2}\right) \text{ for all } x.
\end{aligned}
$$

6. Let

$$\mathbf{b}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

a. Verify that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

b. Find the $\mathcal{B}$-coordinates of $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

7. Complete the following statement of the Fourier convergence theorem.

Theorem. *Suppose that f is a $2p$-periodic* ________________________ *function. The Fourier series of f is given by*

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\boxed{}\right) + b_n \sin\left(\boxed{}\right) \right)$$

where

$$a_0 = \boxed{}$$

and

$$a_n = \boxed{} \qquad b_n = \boxed{}$$

for $n \geq 1$. The Fourier series converges to

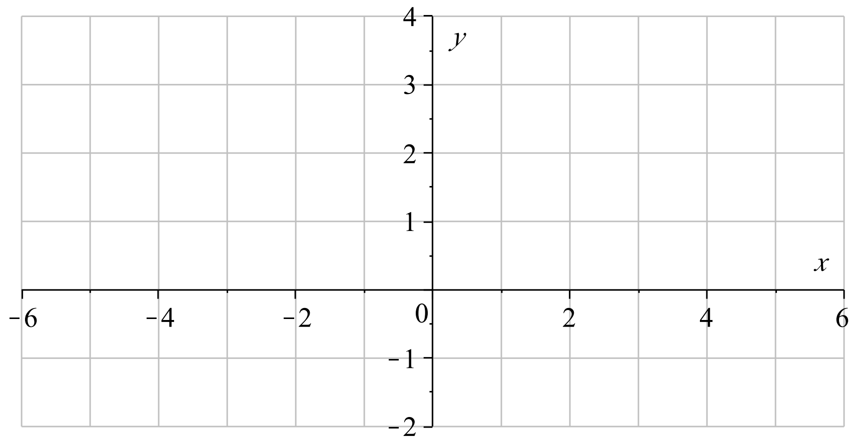
$$\boxed{}$$

for all x.

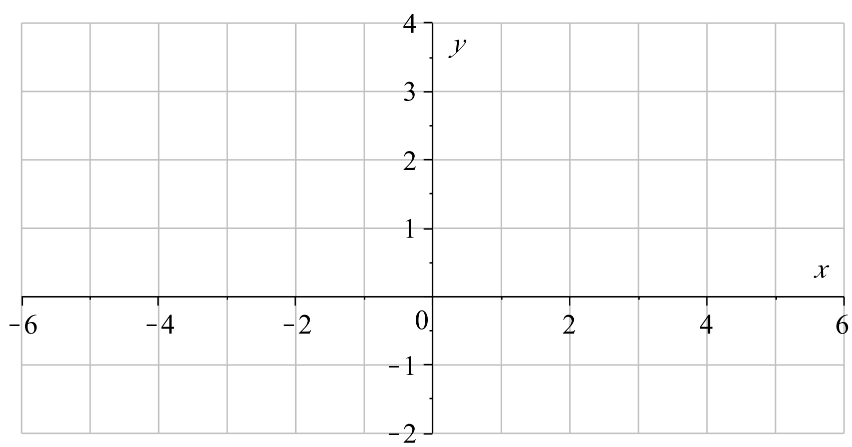
8. Let

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \text{ or } 1 < x < 2, \\ 3 & \text{if } x = 1, \\ 2|x-3| & \text{if } 2 \le x < 4, \\ f(x+4) & \text{otherwise.} \end{cases}$$

a. Carefully sketch the graph of $f(x)$ on the interval $-6 \le x \le 6$.



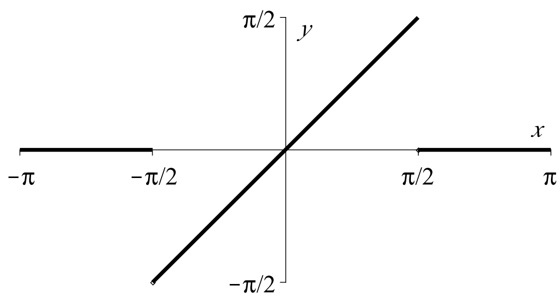
b. Carefully sketch the graph of the Fourier series for $f(x)$ on the interval $-6 \le x \le 6$.



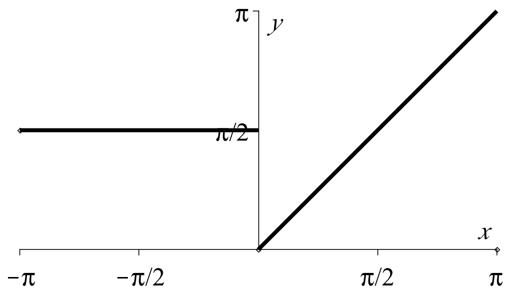
9. Find the Fourier series for the function

$$f(x) = \begin{cases} 1 & \text{if } -1 \leq x < 0, \\ 0 & \text{if } 0 \leq x < 1, \\ f(x+2) & \text{otherwise.} \end{cases}$$

10. Consider the 2π-periodic functions f and g shown below.



$y = f(x)$



$y = g(x)$

If the Fourier series of f is

$$\sum_{n=1}^{\infty} b_n \sin(nx)$$

and the Fourier series of g is

$$A_0 + \sum_{n=1}^{\infty} \left(A_n \cos(nx) + B_n \sin(nx)\right),$$

express A_n and B_n in terms of b_n. You may or may not find it useful to know that

$$\begin{aligned}\cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B),\\ \sin(A+B) &= \sin(A)\cos(B) + \cos(A)\sin(B).\end{aligned}$$