Math 3357 Spring 2012

Partial Differential Equations

Third Midterm Exam

Tuesday, April 24

Your name (please print):

Instructions: This is a closed book, closed notes exam. You may use a calculator, **but the use of other electronic devices such as cell phones, mp3 players, etc. is not permitted.** Unless indicated otherwise, you must justify all of your answers to receive credit. Unjustified answers and/or disorganized or otherwise illegible work will receive partial credit at best. Notation is important, and points will be deducted for incorrect use. Please do all of your work on the paper provided.

The Honor Code requires that you neither give nor receive any aid on this exam.

Please indicate that you have read and understood these guidelines by signing your name in the space provided:

Pledged: ______________________________

Do not write below this line

Problem	**1**	**2**	**3**	**4**	**5**	**6**
Points	10	15	15	20	10	30
Score						

Total:____________________

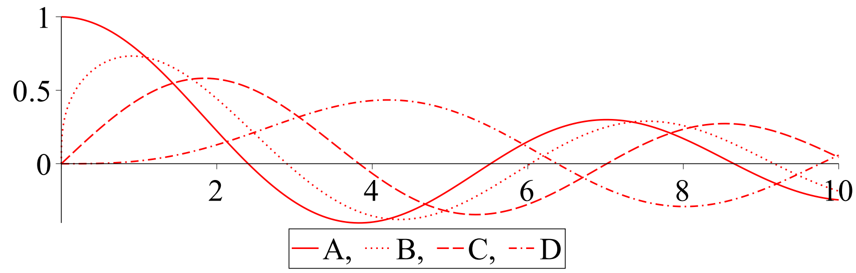
1. Express the differential operator

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$$

in polar coordinates (r, θ). Your final answer may not involve the variables x or y in any way.

2. You do not need to justify your answers to the following four problems.

a. Match the Bessel functions of the first kind, $J_p(x)$, graphed below with their orders.



$p = 1$: ________ $p = 0$: ________ $0 < p < 1$: ________ $p > 1$: ________

b. $y = 2x^3 - 3x$ is an eigenfunction of $y'' - 2xy' + \lambda y = 0$. What is its eigenvalue?

(A) -6 **(B)** 0 **(C)** 6 **(D)** 1 **(E)** None of these

c. Let $y_1 = 1$, $y_2 = x$, $y_3 = 5x^2 - 3$, $y_4 = \sin \pi x$, $y_5 = \cos \pi x$. Which of the following sets of functions are orthogonal on the interval $[-1, 1]$ with respect to the weight $w(x) = x^2$? You may circle more than one choice if necessary.

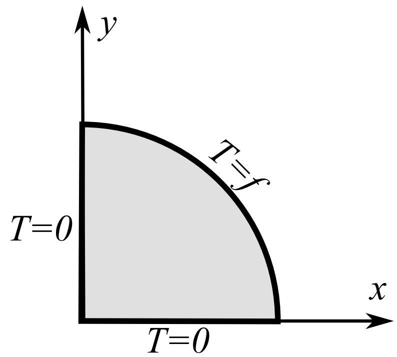
(A) $\{y_1, y_2, y_3\}$ **(B)** $\{y_1, y_4, y_5\}$ **(C)** $\{y_2, y_5\}$ **(D)** $\{y_2, y_3, y_4\}$

d. Which of the following ordinary differential equations could be part of a regular Sturm-Liouville problem on the interval $[0, 3]$? You may circle more than one choice if necessary.

(A) $(x + 1)y'' + 2y' + (x + \lambda)y = 0$ **(B)** $x^2y'' + xy' + (\lambda x^2 - 9)y = 0$

(C) $10y' + (x^2 + 1)\lambda y = 0$ **(D)** $(1 - x)y'' - y' + \lambda y = 0$

3. Consider a metal plate in the shape of a right-angled circular sector with radius a. The top and bottom surfaces of the plate are perfectly insulated, the two straight edges are held at a temperature of zero, end the (time-independent) temperature along the curved edge is given by a function $f(\theta)$, as shown below.



a. Write down a boundary value problem in polar coordinates that describes the steady state temperature distribution in the plate. You may use the fact that the Laplacian in polar coordinates is given by

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.$$

b. Use separation of variables to convert the boundary value problem of part **a** to a system of ordinary differential equations with appropriate boundary conditions. **Do not attempt to solve these equations.**

4. A circular elastic membrane with fixed edges and radius $a = 2$ is stretched to a shape given by

$$f(r, \theta) = \frac{2r^2 - r^3}{10} \cos 5\theta$$

and released at time $t = 0$. As we have seen, the shape of the membrane at any later time t is given by

$$u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mn} r) \; (a_{mn} \cos m\theta + b_{mn} \sin m\theta) \; \cos c\lambda_{mn} t,$$

where J_m is the Bessel function of the first kind of order m, $\lambda_{mn} = \alpha_{mn}/a$, and α_{mn} is the nth positive zero of J_m. The coefficients are given by

$$\begin{aligned}
a_{0n} &= \frac{1}{\pi a^2 J_1^2(\alpha_{0n})} \int_0^{2\pi} \int_0^a f(r, \theta) \, J_0(\lambda_{0n} r) r \, dr \, d\theta, \\
a_{mn} &= \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \int_0^a f(r, \theta) \, \cos m\theta \, J_m(\lambda_{mn} r) r \, dr \, d\theta, \\
b_{mn} &= \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \int_0^a f(r, \theta) \, \sin m\theta \, J_m(\lambda_{mn} r) r \, dr \, d\theta,
\end{aligned}$$

for $m, n = 1, 2, 3, \ldots$.

a. Show that $b_{mn} = 0$ for all m and n.

b. Show that $a_{mn} = 0$ unless $m = 5$.

c. Use parts **a** and **b** to express the shape in terms of the coefficients a_{5n} only. **You do not need to compute these coefficients, nor express them as integrals.**

d. If the same membrane is left flat, but is pushed upward with a uniform velocity of 50 units, one can show that the shape at any later time is given by

$$u(r, \theta, t) = \frac{200}{c} \sum_{n=1}^{\infty} \frac{1}{\alpha_{0n}^2 J_1(\alpha_{0n})} J_0\left(\frac{\alpha_{0n} r}{2}\right) \sin\left(\frac{c\alpha_{0n} t}{2}\right).$$

You do not need to verify this. Use this fact and your answer to part **c** to give an expression for the shape of the membrane if it is both stretched as described above *and* given an initial upward velocity of 50 units at every point.

5. Let $J_p(x)$ denote the Bessel function of the first kind of order p, and let α_{pn} denote the nth positive zero of $J_p(x)$.

a. Given that $\displaystyle\int x^{p+1} J_p(x)\,dx = x^{p+1} J_{p+1}(x) + C$, show that

$$\int x^5 J_2(x)\,dx = x^5 J_3(x) - 2x^4 J_4(x).$$

b. Given that $J_{p+1}(x) + J_{p-1}(x) = \dfrac{2p}{x} J_p(x)$, use part **a** to show that

$$\int_0^{\alpha_{2n}} x^5 J_2(x)\,dx = \alpha_{2n}^3 \left(\alpha_{2n}^2 - 12\right) J_3(\alpha_{2n}).$$

6. Consider the boundary value problem

$$\begin{aligned}
&u_{tt} = x^2 u_{xx} + x u_x, && 1 < x < 2, \quad t > 0,\\
&u(1,t) = u_x(2,t) = 0, && t > 0,\\
&u(x,0) = f(x), && 1 \le x \le 2,\\
&u_t(x,0) = g(x), && 1 \le x \le 2.
\end{aligned}$$

a. Show that separation of variables in this problem yields the system

$$\begin{aligned}
T'' + \lambda T &= 0,\\
x^2 X'' + x X' + \lambda X &= 0, \qquad (1)\\
X(1) = X'(2) &= 0. \qquad (2)
\end{aligned}$$

b. Carefully explain why (1) and (2) constitute a regular Sturm-Liouville problem.

c. One can show that the eigenvalues of (1) and (2) are

$$\lambda_n = \frac{(2n+1)^2\pi^2}{4(\ln 2)^2},$$

with corresponding eigenfunctions

$$X_n = \sin\left(\frac{(2n+1)\pi}{2\ln 2}\ln x\right),$$

for $n = 0, 1, 2, \ldots$. State the orthogonality relations for these functions that are guaranteed by Sturm-Liouville theory.

d. Use part **a** and the information given in **c** to determine the normal modes of the the original PDE boundary value problem.

e. Suppose that $f(x)$ and $g(x)$ are piecewise smooth functions on the interval $[1, 2]$. Use parts **b** – **d** to give the solution to the original PDE boundary value problem. Be sure to give explicit formulas for any coefficients in your solution.