Math 3357 Spring 2012

Partial Differential Equations

Final Exam

Friday, May 4

Your name (please print):

Instructions: This is a closed book, closed notes exam. You may use a calculator, **but the use of other electronic devices such as cell phones, mp3 players, etc. is not permitted.** Unless indicated otherwise, you must justify all of your answers to receive credit. Unjustified answers and/or disorganized or otherwise illegible work will receive partial credit at best. Notation is important, and points will be deducted for incorrect use. Please do all of your work on the paper provided.

The Honor Code requires that you neither give nor receive any aid on this exam.

Please indicate that you have read and understood these guidelines by signing your name in the space provided:

Pledged: ______________________________

Do not write below this line

Problem	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Points	10	10	10	10	20	25	35	30
Score								

Total:____________________

1. Use a linear change of variables to solve the PDE

$$2\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = u.$$

2. For the following functions $f(x)$, sketch the sine and cosine series expansions of $f(x)$ as 2-periodic functions. Make sure your sketch is over the interval $[-3, 3]$.

a. $f(x) = \begin{cases} \frac{1}{2} & \text{if } 0 < x < \frac{1}{2}, \\ 1 - x & \text{if } \frac{1}{2} < x < 1. \end{cases}$

b. $f(x) = x^2$, for $0 < x < 1$.

3. You do not need to justify your answers to the following four problems, and you may circle more than one answer, if needed.

a. Which of the following functions are *already* Fourier series on the interval $[-\pi/2, \pi/2]$?

(A) $\sin(x) + \cos(x)$ **(B)** $1 + \cos(2x) + \cos(4x) + \cos(6x)$ **(C)** $4 + \sin(2\pi x)$

(D) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}$ **(E)** $\displaystyle\frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n \cos(4nx)}{n^2-3}$

b. Assuming that f is a smooth function, which of the following is a solution of

$$xu_x + yu_y = 0?$$

(A) $f(y^2 - x^2)$ **(B)** $f\left(\frac{y^2}{2} - \frac{x^2}{2}\right)$ **(C)** $f\left(\frac{x}{y}\right)$

(D) $f\left(\frac{y}{x}\right)$ **(E)** $f(x^2 + y^2)$ **(F)** None of these

c. Which of the following PDEs are 2nd order, linear and homogeneous?

(A) $u_{xx} + y^2 u_y = 0$ **(B)** $u_{xy} = 2u_x + u_y$ **(C)** $u_{xx} + u_{yy} + x + y = 0$

(D) $u_{yy}^2 + 3u_x = 0$ **(E)** $u_{xx} + u_x u_y - u_{yy} = 0$ **(F)** $u_x^2 + u_x u_y - u_y^2 = 0$

d. Which of the following are steady state solutions of the two-dimensional heat equation?

(A) $e^{-t}(x^2 - y^2)$ **(B)** $x^3 - 3xy^2$ **(C)** $e^x \cos y^2$ **(D)** $\cos x \sinh y$

4. Complete the following statement of the main results of Sturm-Liouville theory.

Theorem. *A regular* Sturm-Liouville problem *on an interval $a < x < b$ consists of an ODE of the form*

$$\boxed{\qquad\qquad}$$

together with boundary conditions of the form

$$\boxed{\qquad\qquad}$$

where the following regularity conditions hold:

i.

ii.

iii.

The eigenvalues of a regular Sturm-Liouville problem are real and form an increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 \cdots$$

that satisfies

$$\lim_{j \to \infty} \lambda_j = \boxed{\qquad}.$$

Moreover, each eigenvalue corresponds to exactly one eigenfunction y_n (up to a scalar multiple), and eigenfunctions with distinct eigenvalues are orthogonal relative to the inner product

$$\boxed{\qquad\qquad}$$

If f is a piecewise smooth function on $a \le x \le b$, then f has an eigenfunction expansion

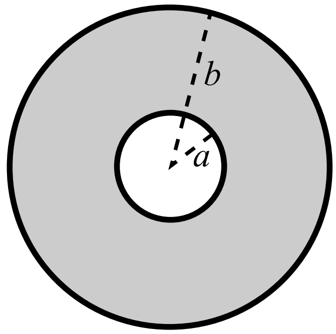
$$\sum_{n=1}^{\infty} A_n \boxed{\qquad}, \text{ where } A_n = \boxed{\qquad\qquad}.$$

The eigenfunction expansion converges to the function

$$\boxed{\qquad\qquad}.$$

5.

Consider an elastic membrane stretched between two concentric circular frames of radii a and b, as shown below.



a. If the membrane is displaced from equilibrium to some initial shape and imparted with an initial velocity, write down a PDE boundary value problem that describes the displacement u of the membrane at any position and any later time. [*Note:* You may find it useful to know that the Laplacian in polar coordinates (r, θ) is given by

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.]$$

b. Use the method of separation of variables to convert the PDE problem from part **a.** to a system of ODEs with corresponding boundary conditions. **Do not attempt to solve these equations.**

6. The right edge of a 3×2 rectangular plate is heated to a temperature described by $T(y) = 7 \sin 5\pi y$, while the remaining three edges are held at a constant temperature of zero.

a. Write down a PDE boundary value problem that models the steady state temperature $u(x, y)$ in the plate at position (x, y).

b. Determine the steady state temperature in the plate. Your final answer may not involve any integrals and must be as simplified as possible.

7. Consider a heated bar of length L that exchanges heat with its surroundings. If we hold the ends of the bar at a constant temperature of 0, then the temperature u of the bar at position x and time t can be modeled by

$$\begin{aligned} &u_t = u_{xx} + 2ku_x, \quad 0 < x < L, \quad t > 0, \\ &u(0,t) = u(L,t) = 0, \quad t > 0, \\ &u(x,0) = f(x), \quad 0 < x < L, \end{aligned}$$

where $k > 0$ is the *coefficient of convection.*

a. Show that separation of variables yields the system

$$X'' + 2kX' + \lambda X = 0, \tag{1}$$
$$X(0) = X(L) = 0, \tag{2}$$
$$T' + \lambda T = 0.$$

b. Show that the equations (1) and (2) defining X constitute a regular Sturm-Liouville problem.

c. Show that there are no eigenvalues of (1) and (2) that satisfy $\lambda \leq k^2$.

d. Show that the eigenfunctions and eigenvalues of (1) and (2) are

$$X_n(x) = e^{-kx} \sin\left(\frac{n\pi x}{L}\right),\, \lambda_n = k^2 + \frac{n^2\pi^2}{L^2},$$

for $n = 1, 2, 3, \ldots$.

e. Give the complete solution to the original PDE boundary value problem.

8. Solve the boundary value problem

$$
\begin{aligned}
&u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \qquad 0 < r < 4, \qquad 0 < z < 5,\\
&u(4, z) = u(r, 0) = 0,\\
&u(r, 5) = f(r).
\end{aligned}
$$

[*Note:* You may find it useful to know that the Sturm-Liouville problem

$$
\begin{aligned}
&(xy')' + \lambda xy = 0, 0 \leq x \leq a,\\
&y(0) \text{ bounded and } y(a) = 0,
\end{aligned}
$$

has eigenvalues $\lambda_n = \mu_n^2$, where $\mu_n = \alpha_n/a$ and α_n is the nth positive zero of the Bessel function J_0, and corresponding eigenfunctions $y_n(x) = J_0(\mu_n x)$.]