PARTIAL DIFFERENTIAL EQUATIONS — ASSIGNMENT 12.1
SPRING 2014 — DUE APRIL 15

Exercise 1. Textbook exercise 4.3.1 (no plot required)

Exercise 2. Textbook exercise 4.3.2 (submit Maple animation throughout TLEARN)

Exercise 3. Textbook exercise 4.3.5 (no plot required)

For $R > 0$ define

$$\delta_R(x,y) = \begin{cases} \frac{1}{\pi R^2} & \text{if } x^2+y^2 \leq R, \\ 0 & \text{otherwise.} \end{cases}$$

The *Dirac delta function* is defined to be the formal limit [1]

$$\delta(x,y) = \lim_{R\to 0+} \delta_R(x,y).$$

Physically, the delta function represents a point impulse at $(0,0)$. According to its definition, for any (continuous) function $f(x,y)$ and any region $\Omega \subset \mathbb{R}^2$ containing $(0,0)$ in its interior, we have

$$\iint_\Omega \delta(x,y) f(x,y)\, dA = \lim_{R\to 0^+} \frac{1}{\pi R^2} \iint_{x^2+y^2\leq R} f(x,y)\, dA = f(0,0),$$

since the quantity inside the limit is the average value of f over the disk $x^2+y^2 \leq R$. To move the delta function's impulse to another point (a,b), we use the function $\delta_{(a,b)}(x,y) = \delta(x-a, y-b)$. A quick change of variables shows that this translated delta function satisfies

$$\iint_\Omega \delta_{(a,b)}(x,y) f(x,y)\, dA = f(a,b). \tag{1}$$

for any region $\Omega \subset \mathbb{R}^2$ containing (a,b) in its interior.

Exercise 4. Solve the vibrating membrane problem if $a = c = 1$, $f(x,y) = 0$ and $g(x,y) = \delta_{(b,0)}(x,y)$, where $0 < b < a$. [*Note:* Formula (1) should make computing the Fourier-Bessel coefficients of the solution very easy.]

Exercise 5. Textbook exercise 6.1.6

Exercise 6. Textbook exercise 6.1.15

[1] This limit is formal only, since it "evaluates" to 0 for $(x,y) \neq (0,0)$ and ∞ for $(x,y) = (0,0)$.