Solving the One-Dimensional Wave Equation Part 2

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Partial Differential Equations
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The 1-D wave equation revisited

Recall: The one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

models the motion of an (ideal) string under tension.

Last time we saw that:

Theorem

The general solution to the wave equation (1) is

$$u(x,t) = F(x+ct) + G(x-ct),$$

where F and G are arbitrary (differentiable) functions of one variable.

Remarks:

- The solution is *uniquely determined* by the initial conditions

$$u(x,0) = f(x), \quad (2)$$
$$u_t(x,0) = g(x). \quad (3)$$

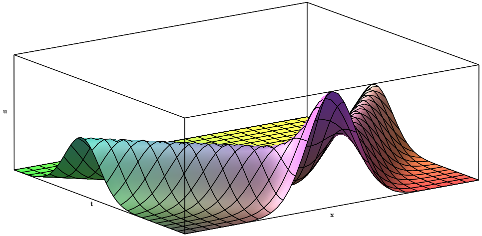
- The domain of $u(x,t)$ is

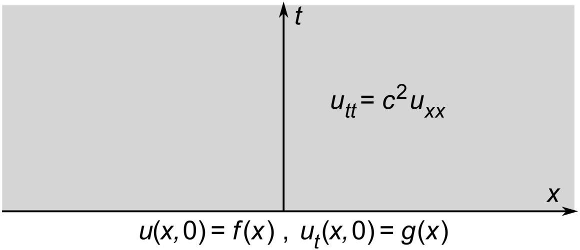
$$R = \mathbb{R} \times [0,\infty).$$

  The function $u(x,t)$ satisfies:
  - $u_{tt} = c^2 u_{xx}$ on the *interior* of R;
  - conditions (2) and (3) on the *boundary* of R.

  This is an example of a *boundary value problem.*

The solution surface and its domain





An additional boundary condition

We now assume that the vibrating string has finite length L, and is fixed at both ends.

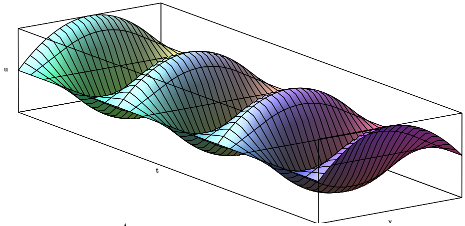
The boundary value problem we now need to consider is

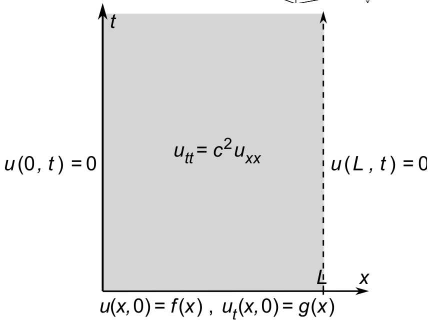
$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, \\ u(0,t) &= u(L,t) = 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x), \end{aligned}$$

on the domain

$$R = [0,L] \times [0,\infty).$$

The solution surface and its domain





D'Alembert's solution of the vibrating string problem

We now turn to the solution of the (finite) vibrating string problem.

We would like to apply the general solution

$$u(x,t) = F(x+ct) + G(x-ct).$$

Problem: The initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ only apply for

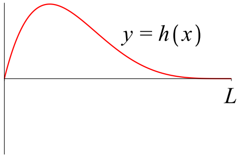
$$0 \le x \le L,$$

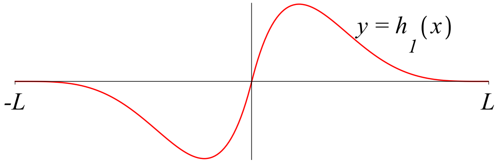
i.e. along the length of the string. But determining F and G requires initial data for *all* $x \in \mathbb{R}$.

Idea: Extend f and g (in some particular way) to all of $\mathbb{R}$.

Periodic extensions

Given a function $h(x)$ with domain $[0, L]$ we first extend it to an odd function $h_1(x)$ on $[-L, L]$ by reflecting its graph through the origin:

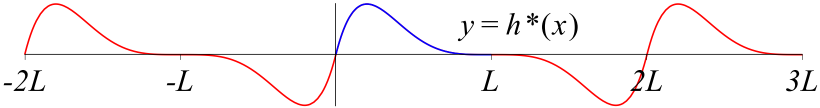




Symbolically:

$$h_1(x) = \begin{cases} h(x) & \text{if } 0 < x \leq L, \\ 0 & \text{if } x = 0, \\ -h(-x) & \text{if } -L \leq x < 0. \end{cases}$$

We then extend $h_1(x)$ to a function $h^*(x)$ on all of $\mathbb{R}$ by repeatedly "cutting and pasting" its graph:



This is called the $2L$*-periodic odd extension* of $h(x)$. Symbolically:

$$h^*(x) = h_1\left(x - 2L\left[\frac{x+L}{2L}\right]\right),$$

where $[\cdot]$ is the floor function.

Back to the vibrating string

Goal: Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ on the domain $[0, L] \times [0, \infty)$, subject to the boundary conditions

$$u(0,t) = u(L,t) = 0,$$
$$u(x,0) = f(x),\ u_t(x,0) = g(x).$$

Solution: We first use the $2L$-periodic extensions of f and g and solve the boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$
$$u(x,0) = f^*(x),\ u_t(x,0) = g^*(x),$$

on $\mathbb{R} \times [0, \infty)$. Then we show that for $0 \leq x \leq L$, this $u(x,t)$ solves the vibrating string problem.

For $0 \leq x \leq L$, we immediately have

$$u(x,0) = f^*(x) = f(x),$$
$$u_t(x,0) = g^*(x) = g(x).$$

To verify the other boundary conditions, we write

$$u(x,t) = F(x+ct) + G(x-ct)$$

and solve for F and G. We find that

$$\begin{aligned} f^*(x) &= u(x,0) = F(x) + G(x), \\ (f^*)'(x) &= F'(x) + G'(x), \\ g^*(x) &= u_t(x,0) = cF'(x) - cG'(x). \end{aligned}$$

The last two equations are equivalent to

$$\begin{pmatrix} 1 & 1 \\ c & -c \end{pmatrix} \begin{pmatrix} F' \\ G' \end{pmatrix} = \begin{pmatrix} (f^*)' \\ g^* \end{pmatrix}.$$

Matrix inversion gives

$$\begin{pmatrix} F' \\ G' \end{pmatrix} = \frac{-1}{2c} \begin{pmatrix} -c & -1 \\ -c & 1 \end{pmatrix} \begin{pmatrix} (f^*)' \\ g^* \end{pmatrix} = \begin{pmatrix} \dfrac{(f^*)'}{2} + \dfrac{g^*}{2c} \\[2ex] \dfrac{(f^*)'}{2} - \dfrac{g^*}{2c} \end{pmatrix}$$

Therefore, by FTOC,

$$\begin{aligned} F(x+ct) - F(0) &= \int_0^{x+ct} F'(s)\,ds = \int_0^{x+ct} \frac{(f^*)'(s)}{2} + \frac{g^*(s)}{2c}\,ds \\ &= \frac{1}{2}(f^*(x+ct) - f^*(0)) + \frac{1}{2c}\int_0^{x+ct} g^*(s)\,ds. \end{aligned}$$

Likewise, one can show

$$\begin{aligned} G(x-ct)-G(0) &= \frac{1}{2}(f^*(x-ct)-f^*(0)) - \frac{1}{2c}\int_0^{x-ct} g^*(s)\,ds \\ &= \frac{1}{2}(f^*(x-ct)-f^*(0)) + \frac{1}{2c}\int_{x-ct}^{0} g^*(s)\,ds. \end{aligned}$$

Since $f^*(0)=0$ and $f^*(x)=F(x)+G(x)$, it now follows that

$$\begin{aligned} u(x,t) &= F(x+ct)+G(x-ct) \\ &= F(0)+G(0)+\frac{f^*(x+ct)+f^*(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds \\ &= \frac{f^*(x+ct)+f^*(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds. \end{aligned}$$

It remains to show that $u(0,t) = u(L,t) = 0$ for all $t > 0$.

Setting $x = 0$ in the expression above yields

$$u(0,t) = \frac{f^*(ct) + f^*(-ct)}{2} + \frac{1}{2c}\int_{-ct}^{ct} g^*(s)\, ds = 0$$

since f^* and g^* are both odd functions.

Setting $x = L$ we get

$$u(L,t) = \underbrace{\frac{f^*(L+ct) + f^*(L-ct)}{2}}_{A} + \underbrace{\frac{1}{2c}\int_{L-ct}^{L+ct} g^*(s)\, ds}_{B}.$$

Because f^* and g^* are both $2L$-periodic and odd, one can show that $A = B = 0$ (HW), which finishes our work.

Summary

Theorem (D'Alembert)

The solution of the vibrating string problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$
$$u(0,t) = u(L,t) = 0,$$
$$u(x,0) = f(x),\ u_t(x,0) = g(x).$$

on the domain $[0,L] \times [0,\infty)$ *is given by*

$$u(x,t) = \frac{f^*(x+ct) + f^*(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds,$$

where f^* *and* g^* *are the* $2L$*-periodic odd extensions of* f *and* g.

Remark: One can show that, in fact, this solution is *unique*.

Remarks

- When $g \equiv 0$, the solution consists of two superimposed traveling waves, both with the same initial shape, moving in opposite directions.
- In general, if $G(x)$ is an antiderivative of $g^*(x)$, then

$$\int_{x-ct}^{x+ct} g^*(s)\,ds = G(x+ct) - G(x-ct)$$

so that

$$u(x,t) = \left(\frac{f^*(x+ct)}{2} + \frac{G(x+ct)}{2c}\right) + \left(\frac{f^*(x-ct)}{2} - \frac{G(x-ct)}{2c}\right),$$

i.e. $u(x,t)$ is a superposition of two *different* oppositely moving traveling waves.

Example

Show that the solution to the vibrating string problem is periodic in time, with period $2L/c$. That is, show that if $u(x,t)$ is a solution, then

$$u(x, t + 2L/c) = u(x,t).$$

First, if a function h has period $2L$, we have

$$h(x \pm c(t + 2L/c)) = h(x \pm ct \pm 2L) = h(x \pm ct),$$

which shows that $h(x \pm ct)$ has period $2L/c$ in t.

The solution $u(x,t)$ is built of functions of the form $h(x \pm ct)$, with $h = f^*, G$.

So, it suffices to show that f^* and G have period $2L$.

By definition, f^* has period $2L$.

According to the FTOC

$$G(x+2L) - G(x) = \int_x^{x+2L} g^*(s)\,ds = \int_{-L}^{L} g^*(s)\,ds = 0,$$

since g^* is $2L$-periodic and odd. This shows

$$G(x+2L) = G(x),$$

which is what we wanted to show.

Remark: The fact that $G(x)$ is $2L$-periodic is independently useful.

Example

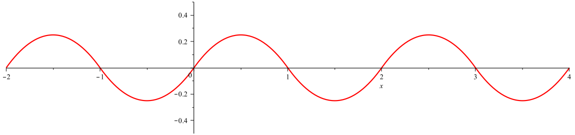
Solve the vibrating string problem with $L = c = 1$, $f(x) = x(1-x)$ and $g(x) = 1 - x$.

We first find $f^*(x)$. The odd extension of f to $[-1,1]$ is

$$f_1(x) = \begin{cases} x(x+1) & \text{if } -1 \leq x < 0, \\ 0 & \text{if } x = 0, \\ x(x-1) & \text{if } 0 < x \leq 1. \end{cases}$$

Hence

$$f^*(x) = f_1\left(x - 2\left[\frac{x+1}{2}\right]\right).$$



The odd extension of g to $[-1,1]$ is

$$g_1(x) = \begin{cases} -1-x & \text{if } -1 \le x < 0, \\ 0 & \text{if } x = 0, \\ 1-x & \text{if } 0 < x \le 1. \end{cases} .$$

Now we need an antiderivative of g_1. For $x \in [-1,0]$ we have

$$G_1(x) = \int_{-1}^{x} g_1(s)\,ds = \int_{-1}^{x} -1-s\,ds = -\frac{x^2}{2} - x - \frac{1}{2},$$

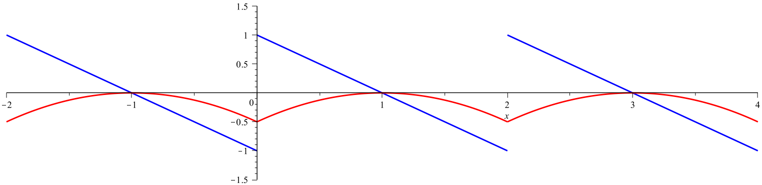
and for $x \in [0,1]$ we have

$$G_1(x) = \int_{-1}^{x} g_1(s)\,ds = \int_{-1}^{0} g_1(s)\,ds + \int_{0}^{x} 1-s\,ds = -\frac{x^2}{2} + x - \frac{1}{2}.$$

The function G is then the 2-periodic extension of G_1:

$$G(x) = G_1\left(x - 2\left[\frac{x+1}{2}\right]\right).$$

Here are the graphs of g^* (in blue) and G (in red):



Since $c = 1$, the solution is then

$$u(x,t) = \frac{f^*(x+t) + G(x+t)}{2} + \frac{f^*(x-t) - G(x-t)}{2}.$$

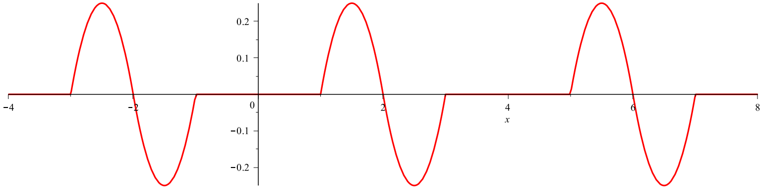
Example

A string with $L = 2$ and $c = 3$ is given the initial shape

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1, \\ (x-1)(2-x) & \text{if } 1 < x \le 2 \end{cases}$$

and is released with zero initial velocity. How long does it take before the point $x = \frac{1}{5}$ begins to vibrate?

First, let's look at the graph of $f^*(x)$.



Since $g \equiv 0$, the solution $u(x,t)$ is a superposition two copies of f^*, one moving left, the other right, with speed $c = 3$.

The graph shows that the left-moving copy reaches $x = \frac{1}{5}$ first.

The vibration must move $1 - \frac{1}{5} = \frac{4}{5}$ of a unit to reach $x = \frac{1}{5}$.

Thus, the amount of time it takes for this to happen is

$$t = \frac{4/5}{3} = \frac{4}{15}.$$