Linear PDEs and the Principle of Superposition

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Partial Differential Equations
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Linear differential operators

Definition: A *linear differential operator* (in the variables $x_1, x_2, \ldots x_n$) is a sum of terms of the form

$$A(x_1, x_2, \ldots, x_n) \frac{\partial^{a_1+a_2+\cdots+a_n}}{\partial x_1^{a_1} \partial x_2^{a_2} \cdots \partial x_n^{a_n}},$$

where each $a_i \geq 0$.

Examples: The following are linear differential operators.

1. The Laplacian:

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

2. $W = c^2 \nabla^2 - \dfrac{\partial^2}{\partial t^2}$

3. $H = c^2\nabla^2 - \dfrac{\partial}{\partial t}$

4. $T = \dfrac{\partial}{\partial t} - v_1\dfrac{\partial}{\partial x_1} - v_2\dfrac{\partial}{\partial x_2} - \cdots - v_n\dfrac{\partial}{\partial x_n} = \dfrac{\partial}{\partial t} - \mathbf{v}\cdot\nabla$

5. The general first order linear operator (in two variables):

$$D_1 = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y} + C(x,y)$$

6. The general second order linear operator (in two variables):

$$D_2 = A(x,y)\frac{\partial^2}{\partial x^2} + 2B(x,y)\frac{\partial^2}{\partial x\partial y} + C(x,y)\frac{\partial^2}{\partial y^2} + D(x,y)\frac{\partial}{\partial x} + E(x,y)\frac{\partial}{\partial y} + F(x,y)$$

Theorem

If D is a linear differential operator (in the variables $x_1, x_2, \cdots x_n$), u_1 and u_2 are functions (in the same variables), and c_1 and c_2 are constants, then

$$D(c_1 u_1 + c_2 u_2) = c_1 D u_1 + c_2 D u_2.$$

Remarks:

- This follows immediately from the fact that each partial derivative making up L has this property, e.g.

$$\frac{\partial^3}{\partial x_1^2 \partial x_2}(c_1 u_1 + c_2 u_2) = c_1 \frac{\partial^3 u_1}{\partial x_1^2 \partial x_2} + c_2 \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2}.$$

- This property extends (in the obvious way) to any number of functions and constants.

Linear PDEs

Definition: A *linear PDE* (in the variables $x_1, x_2, \cdots, x_n$) has the form

$$Du = f \tag{1}$$

where:

- D is a linear differential operator (in $x_1, x_2, \cdots, x_n$),
- f is a function (of $x_1, x_2, \cdots, x_n$).

We say that (1) is *homogeneous* if $f \equiv 0$.

Examples: The following are examples of linear PDEs.

1. The Lapace equation: $\nabla^2 u = 0$ (homogeneous)
2. The wave equation: $c^2 \nabla^2 u - \dfrac{\partial^2 u}{\partial t^2} = 0$ (homogeneous)

3. The heat equation: $c^2\nabla^2 u - \dfrac{\partial u}{\partial t} = 0$ (homogeneous)
4. The Poisson equation: $\nabla^2 u = f(x_1, x_2, \ldots, x_n)$ (inhomogeneous if $f \not\equiv 0$)
5. The advection equation: $\dfrac{\partial u}{\partial t} + \kappa\dfrac{\partial u}{\partial x} + ru = k(x,t)$ (inhomogeneous if $k \not\equiv 0$)
6. The telegraph equation: $\dfrac{\partial^2 u}{\partial t^2} + 2B\dfrac{\partial u}{\partial t} - c^2\dfrac{\partial^2 u}{\partial x^2} + Au = 0$ (homogeneous)

Non-examples: The following are non-linear PDEs (why?).

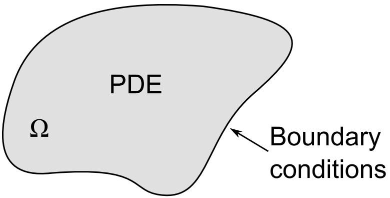
1. The Liouville equation: $\nabla^2 u + e^{\lambda u} = 0$
2. The KdV equation: $\dfrac{\partial u}{\partial t} + \dfrac{\partial^3 u}{\partial x^3} - 6u\dfrac{\partial u}{\partial x} = 0$

Linear boundary conditions

A *boundary value problem* (BVP) consists of:

- a domain $\Omega \subseteq \mathbb{R}^n$,
- a PDE (in n independent variables) to be solved in the interior of Ω,
- a collection of *boundary conditions* to be satisfied on the boundary of Ω.

The data for a BVP:



Definition: Let $\Omega \subseteq \mathbb{R}^n$ be the domain of a BVP and let A be a subset of the boundary of Ω.

We say that a boundary condition on A is *linear* if it has the form

$$Du\big|_A = f\big|_A \tag{2}$$

where:

- D is a linear differential operator (in $x_1, x_2, \cdots, x_n$),
- f is a function (of $x_1, x_2, \cdots, x_n$).

(The notation $\cdot\big|_A$ means "restricted to A.") We say that (2) is *homogeneous* if $f \equiv 0$.

Examples: The following are linear boundary conditions.

1. Dirichlet conditions: $u\big|_A = f\big|_A$, such as

$$u(x,0) = f(x) \text{ for } 0 < x < L, \text{ or } u(L,t) = 0 \text{ for } t > 0$$

2. Neumann conditions: $\left.\dfrac{\partial u}{\partial \mathbf{n}}\right|_A = f\big|_A$, where $\dfrac{\partial u}{\partial \mathbf{n}}$ is the directional derivative perpendicular to A, such as

$$u_t(x,0) = g(x) \text{ for } 0 < x < L, \text{ or } u_x(0,t) = 0 \text{ for } t > 0$$

3. Robin conditions: $\left. u + a\dfrac{\partial u}{\partial \mathbf{n}}\right|_A = f\big|_A$, such as

$$u(L,t) + u_x(L,t) = 0 \text{ for } t > 0$$

The principle of superposition

Theorem

Let D be a linear differential operator (in the variables $x_1, x_2, \dots, x_n$), let f_1 and f_2 be functions (in the same variables), and let c_1 and c_2 be constants.

- *If u_1 solves the linear PDE $Du = f_1$ and u_2 solves $Du = f_2$, then $u = c_1 u_1 + c_2 u_2$ solves $Du = c_1 f_1 + c_2 f_2$. In particular, if u_1 and u_2 both solve the same homogeneous linear PDE, so does $u = c_1 u_1 + c_2 u_2$.*
- *If u_1 satisfies the linear boundary condition $Du\big|_A = f_1\big|_A$ and u_2 satisfies $Du\big|_A = f_2\big|_A$, then $u = c_1 u_1 + c_2 u_2$ satisfies $Du\big|_A = c_1 f_1 + c_2 f_2\big|_A$. In particular, if u_1 and u_2 both satisfy the same homogeneous linear boundary condition, so does $u = c_1 u_1 + c_2 u_2$.*

Remarks

The superposition principle:

- Holds because of the linearity of D, e.g. if $Du_1 = f_1$ and $Du_2 = f_2$, then

$$D(c_1 u_1 + c_2 u_2) = c_1 D u_1 + c_2 D u_2 = c_1 f_1 + c_2 f_2.$$

- Extends (in the obvious way) to any number of functions and constants.
- Says that linear combinations of solutions to a linear PDE yield more solutions.
- Says that linear combinations of functions satisfying linear boundary conditions yield functions that satisfy the corresponding combination of boundary conditions.

Example

Consider the boundary value problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \; y > 0, \\ u(x,0) &= 0, \; -\infty < x < \infty. \end{aligned}$$

The functions

$$\begin{aligned} u_1(x,y) &= \cos(x)(e^y - e^{-y}), \\ u_2(x,y) &= \sin(y)(e^x + e^{-x}) \end{aligned}$$

are both solutions.
Since the PDE and boundary conditions are both linear and homogeneous,

$$u = c_1 u_1 + c_2 u_2 = c_1 \cos(x)(e^y - e^{-y}) + c_2 \sin(y)(e^x + e^{-x})$$

solve the same problem, for any constants c_1 and c_2.

Example

Consider the vibrating string problem with initial data

$$u(x,0) = \sin\left(\frac{\pi x}{L}\right),\ u_t(x,0) = 0,\ 0 \le x \le L.$$

One can easily check that

$$u_1(x,t) = \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{c\pi t}{L}\right)$$

is a solution to this problem. If we change the initial conditions to

$$u(x,0) = 0,\ u_t(x,0) = \sin\left(\frac{\pi x}{L}\right),\ 0 \le x \le L,$$

then

$$u_2(x,t) = \frac{L}{c\pi}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{c\pi t}{L}\right)$$

is a solution.

Since the wave equation and all of the boundary conditions in the vibrating string problem are linear, it follows immediately that

$$u = 2u_1 - u_2 = \sin\left(\frac{\pi x}{L}\right)\left(2\cos\left(\frac{c\pi t}{L}\right) - \frac{L}{c\pi}\sin\left(\frac{c\pi t}{L}\right)\right)$$

solves the vibrating string problem with the initial conditions

$$u(x,0) = 2\sin\left(\frac{\pi x}{L}\right),\ u_t(x,0) = -\sin\left(\frac{\pi x}{L}\right).$$

Non-example

Warning: The principle of superposition can *easily* fail for nonlinear PDEs or boundary conditions.

Consider the nonlinear PDE

$$u_x + u^2 u_y = 0.$$

One solution of this PDE is

$$u_1(x, y) = \frac{-1 + \sqrt{1 + 4xy}}{2x}.$$

However, the function $u = cu_1$ *does not* solve the same PDE unless $c = 0, \pm 1$.

Example

More generally, for $n = 1, 2, 3, \ldots$ the functions

$$u_n(x,t) = \sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right),$$
$$v_n(x,t) = \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right),$$

solve the vibrating string problem with initial conditions

$$u_n(x,0) = 0,$$
$$(u_n)_t(x,0) = \frac{n\pi c}{L}\sin\left(\frac{n\pi x}{L}\right),$$
$$v_n(x,0) = \sin\left(\frac{n\pi x}{L}\right),$$
$$(v_n)_t(x,0) = 0.$$

By the principle of superposition, it follows that the function

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \\ &= \sum_{n=1}^{\infty}\left(a_n \sin\left(\frac{n\pi ct}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

solves the vibrating string problem with initial conditions

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \\ u_t(x,0) &= \sum_{n=1}^{\infty} a_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right). \end{aligned}$$

The moral

One can solve the vibrating string problem with initial conditions

$$u(x,0) = f(x),\ u_t(x,0) = g(x), 0 \le x \le L,$$

provided that $f(x)$ and $g(x)$ can be expressed as (possibly infinite) linear combinations of the functions $\sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, 3, \ldots$.

Such combinations are examples of *Fourier series*.

Questions:

- Which functions are expressible as Fourier series?
- How can we find the Fourier series expansion of a given function?

We'll begin to answer these questions next week!