Linear differential operators

**Definition:** A *linear differential operator* (in the variables $x_1, x_2, \ldots, x_n$) is a sum of terms of the form

$$A(x_1, x_2, \ldots, x_n) \frac{\partial^{a_1+a_2+\cdots+a_n}}{\partial x_1^{a_1} \partial x_2^{a_2} \cdots \partial x_n^{a_n}},$$

where each $a_i \geq 0$.

**Examples:** The following are linear differential operators.

1. The Laplacian:

   $$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

2. $W = c^2 \nabla^2 - \frac{\partial^2}{\partial t^2}$
3. \[ H = c^2 \nabla^2 - \frac{\partial}{\partial t} \]

4. \[ T = \frac{\partial}{\partial t} - v_1 \frac{\partial}{\partial x_1} - v_2 \frac{\partial}{\partial x_2} - \cdots - v_n \frac{\partial}{\partial x_n} = \frac{\partial}{\partial t} - \mathbf{v} \cdot \nabla \]

5. The general first order linear operator (in two variables):

\[ D_1 = A(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y} + C(x, y) \]

6. The general second order linear operator (in two variables):

\[ D_2 = A(x, y) \frac{\partial^2}{\partial x^2} + 2B(x, y) \frac{\partial^2}{\partial x \partial y} + C(x, y) \frac{\partial^2}{\partial y^2} \]

\[ + D(x, y) \frac{\partial}{\partial x} + E(x, y) \frac{\partial}{\partial y} + F(x, y) \]
Theorem

If $D$ is a linear differential operator (in the variables $x_1, x_2, \cdots x_n$), $u_1$ and $u_2$ are functions (in the same variables), and $c_1$ and $c_2$ are constants, then

$$D(c_1 u_1 + c_2 u_2) = c_1 Du_1 + c_2 Du_2.$$  

Remarks:

- This follows immediately from the fact that each partial derivative making up $L$ has this property, e.g.

  $$\frac{\partial^3}{\partial x_1^2 \partial x_2} (c_1 u_1 + c_2 u_2) = c_1 \frac{\partial^3 u_1}{\partial x_1^2 \partial x_2} + c_2 \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2}.$$  

- This property extends (in the obvious way) to any number of functions and constants.
Definition: A linear PDE (in the variables $x_1, x_2, \cdots, x_n$) has the form
\[ Du = f \] (1)
where:
- $D$ is a linear differential operator (in $x_1, x_2, \cdots, x_n$),
- $f$ is a function (of $x_1, x_2, \cdots, x_n$).

We say that (1) is homogeneous if $f \equiv 0$.

Examples: The following are examples of linear PDEs.

1. The Laplace equation: $\nabla^2 u = 0$ (homogeneous)
2. The wave equation: $c^2 \nabla^2 u - \frac{\partial^2 u}{\partial t^2} = 0$ (homogeneous)
3. The heat equation: \( c^2 \nabla^2 u - \frac{\partial u}{\partial t} = 0 \) (homogeneous)

4. The Poisson equation: \( \nabla^2 u = f(x_1, x_2, \ldots, x_n) \)
   (inhomogeneous if \( f \neq 0 \))

5. The advection equation: \( \frac{\partial u}{\partial t} + \kappa \frac{\partial u}{\partial x} + ru = k(x, t) \)
   (inhomogeneous if \( k \neq 0 \))

6. The telegraph equation: \( \frac{\partial^2 u}{\partial t^2} + 2B \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} + Au = 0 \)
   (homogeneous)

**Non-examples:** The following are non-linear PDEs (why?).

1. The Liouville equation: \( \nabla^2 u + e^{\lambda u} = 0 \)

2. The KdV equation: \( \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 6u \frac{\partial u}{\partial x} = 0 \)
Linear boundary conditions

A boundary value problem (BVP) consists of:

- a domain $\Omega \subseteq \mathbb{R}^n$,
- a PDE (in $n$ independent variables) to be solved in the interior of $\Omega$,
- a collection of boundary conditions to be satisfied on the boundary of $\Omega$.

The data for a BVP:

![Diagram of a domain $\Omega$ with a PDE in the interior and boundary conditions on the boundary.]
**Definition:** Let \( \Omega \subseteq \mathbb{R}^n \) be the domain of a BVP and let \( A \) be a subset of the boundary of \( \Omega \).

We say that a boundary condition on \( A \) is *linear* if it has the form

\[
Du|_A = f|_A
\]  \hspace{1cm} (2)

where:
- \( D \) is a linear differential operator (in \( x_1, x_2, \ldots, x_n \)),
- \( f \) is a function (of \( x_1, x_2, \ldots, x_n \)).

(The notation \( \cdot|_A \) means “restricted to \( A \).”) We say that (2) is *homogeneous* if \( f \equiv 0 \).
Examples: The following are linear boundary conditions.

1. Dirichlet conditions: $u|_A = f|_A$, such as

   $$u(x,0) = f(x) \text{ for } 0 < x < L, \text{ or } u(L,t) = 0 \text{ for } t > 0$$

2. Neumann conditions: $\frac{\partial u}{\partial n}|_A = f|_A$, where $\frac{\partial u}{\partial n}$ is the directional derivative perpendicular to $A$, such as

   $$u_t(x,0) = g(x) \text{ for } 0 < x < L, \text{ or } u_x(0,t) = 0 \text{ for } t > 0$$

3. Robin conditions: $u + a \frac{\partial u}{\partial n}|_A = f|_A$, such as

   $$u(L,t) + u_x(L,t) = 0 \text{ for } t > 0$$
The principle of superposition

**Theorem**

Let $D$ be a linear differential operator (in the variables $x_1, x_2, \ldots, x_n$), let $f_1$ and $f_2$ be functions (in the same variables), and let $c_1$ and $c_2$ be constants.

- If $u_1$ solves the linear PDE $Du = f_1$ and $u_2$ solves $Du = f_2$, then $u = c_1 u_1 + c_2 u_2$ solves $Du = c_1 f_1 + c_2 f_2$. In particular, if $u_1$ and $u_2$ both solve the same homogeneous linear PDE, so does $u = c_1 u_1 + c_2 u_2$.

- If $u_1$ satisfies the linear boundary condition $Du|_A = f_1|_A$ and $u_2$ satisfies $Du|_A = f_2|_A$, then $u = c_1 u_1 + c_2 u_2$ satisfies $Du|_A = c_1 f_1 + c_2 f_2|_A$. In particular, if $u_1$ and $u_2$ both satisfy the same homogeneous linear boundary condition, so does $u = c_1 u_1 + c_2 u_2$. 
The superposition principle:

- Holds because of the linearity of $D$, e.g. if $Du_1 = f_1$ and $Du_2 = f_2$, then

$$D(c_1 u_1 + c_2 u_2) = c_1 Du_1 + c_2 Du_2 = c_1 f_1 + c_2 f_2.$$  

- Extends (in the obvious way) to any number of functions and constants.

- Says that linear combinations of solutions to a linear PDE yield more solutions.

- Says that linear combinations of functions satisfying linear boundary conditions yield functions that satisfy the corresponding combination of boundary conditions.
Example

Consider the boundary value problem

\[ u_{xx} + u_{yy} = 0, \quad y > 0, \]
\[ u(x, 0) = 0, \quad -\infty < x < \infty. \]

The functions

\[ u_1(x, y) = \cos(x)(e^y - e^{-y}), \]
\[ u_2(x, y) = \sin(y)(e^x + e^{-x}) \]

are both solutions. Since the PDE and boundary conditions are both linear and homogeneous,

\[ u = c_1 u_1 + c_2 u_2 = c_1 \cos(x)(e^y - e^{-y}) + c_2 \sin(y)(e^x + e^{-x}) \]

solve the same problem, for any constants \( c_1 \) and \( c_2 \).
Example

Consider the vibrating string problem with initial data

\[ u(x, 0) = \sin \left( \frac{\pi x}{L} \right), \quad u_t(x, 0) = 0, \quad 0 \leq x \leq L. \]

One can easily check that

\[ u_1(x, t) = \sin \left( \frac{\pi x}{L} \right) \cos \left( \frac{c \pi t}{L} \right) \]

is a solution to this problem. If we change the initial conditions to

\[ u(x, 0) = 0, \quad u_t(x, 0) = \sin \left( \frac{\pi x}{L} \right), \quad 0 \leq x \leq L, \]

then

\[ u_2(x, t) = \frac{L}{c \pi} \sin \left( \frac{\pi x}{L} \right) \sin \left( \frac{c \pi t}{L} \right) \]

is a solution.
Since the wave equation and all of the boundary conditions in the vibrating string problem are linear, it follows immediately that

\[ u = 2u_1 - u_2 = \sin \left( \frac{\pi x}{L} \right) \left( 2 \cos \left( \frac{c \pi t}{L} \right) - \frac{L}{c \pi} \sin \left( \frac{c \pi t}{L} \right) \right) \]

solves the vibrating string problem with the initial conditions

\[ u(x, 0) = 2 \sin \left( \frac{\pi x}{L} \right), \quad u_t(x, 0) = -\sin \left( \frac{\pi x}{L} \right). \]
Warning: The principle of superposition can *easily* fail for nonlinear PDEs or boundary conditions.

Consider the nonlinear PDE

\[ u_x + u^2 u_y = 0. \]

One solution of this PDE is

\[ u_1(x, y) = \frac{-1 + \sqrt{1 + 4xy}}{2x}. \]

However, the function \( u = cu_1 \) *does not* solve the same PDE unless \( c = 0, \pm 1 \).
More generally, for \( n = 1, 2, 3, \ldots \) the functions

\[
\begin{align*}
    u_n(x, t) &= \sin \left( \frac{n\pi ct}{L} \right) \sin \left( \frac{n\pi x}{L} \right), \\
    v_n(x, t) &= \cos \left( \frac{n\pi ct}{L} \right) \sin \left( \frac{n\pi x}{L} \right),
\end{align*}
\]

solve the vibrating string problem with initial conditions

\[
\begin{align*}
    u_n(x, 0) &= 0, \\
    (u_n)_t(x, 0) &= \frac{n\pi c}{L} \sin \left( \frac{n\pi x}{L} \right), \\
    v_n(x, 0) &= \sin \left( \frac{n\pi x}{L} \right), \\
    (v_n)_t(x, 0) &= 0.
\end{align*}
\]
By the principle of superposition, it follows that the function

\[ u(x, t) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{n\pi c t}{L} \right) \sin \left( \frac{n\pi x}{L} \right) + b_n \cos \left( \frac{n\pi c t}{L} \right) \sin \left( \frac{n\pi x}{L} \right) \]

solves the vibrating string problem with initial conditions

\[ u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right), \]
\[ u_t(x, 0) = \sum_{n=1}^{\infty} a_n \frac{n\pi c}{L} \sin \left( \frac{n\pi x}{L} \right). \]
The moral

One can solve the vibrating string problem with initial conditions

\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq L, \]

provided that \( f(x) \) and \( g(x) \) can be expressed as (possibly infinite) linear combinations of the functions

\[ \sin \left( \frac{n\pi x}{L} \right), \quad n = 1, 2, 3, \ldots \]

Such combinations are examples of *Fourier series*.

**Questions:**

- Which functions are expressible as Fourier series?
- How can we find the Fourier series expansion of a given function?

We’ll begin to answer these questions next week!