The One-Dimensional Heat Equation

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Introduction

Goal: Model heat flow in a one-dimensional object (thin rod).

Set up: Place rod of length L along x-axis, one end at origin:

Let $u(x, t) =$ temperature in rod at position x, time t.

(Ideal) Assumptions:

- Rod is given some initial temperature distribution $f(x)$ along its length.
- Rod is perfectly insulated, i.e. heat only moves horizontally.
- No internal heat sources or sinks.

The Heat Equation

One can show that u satisfies the *one-dimensional heat equation*

$$u_t = c^2 u_{xx}.$$

Remarks:

- This can be derived via conservation of energy and Fourier's law of heat conduction (see textbook pp. 143-144).
- The constant c^2 is the *thermal diffusivity*:

$$c^2 = \frac{K_0}{s\rho}, \qquad \begin{aligned} K_0 &= \text{ thermal conductivity,} \\ s &= \text{ specific heat,} \\ \rho &= \text{ density.} \end{aligned}$$

Initial and Boundary Conditions

To completely determine u we must also specify:

Initial conditions: The initial temperature profile

$$u(x,0) = f(x) \text{ for } 0 < x < L.$$

Boundary conditions: Specific behavior at $x_0 = 0, L$:

1. Constant temperature: $u(x_0, t) = T$ for $t > 0$.
2. Insulated end: $u_x(x_0, t) = 0$ for $t > 0$.
3. Radiating end: $u_x(x_0, t) = Au(x_0, t)$ for $t > 0$.

Solving the Heat Equation

Case 1: homogeneous Dirichlet boundary conditions

We now apply separation of variables to the heat problem

$$\begin{aligned} u_t &= c^2 u_{xx} && (0 < x < L, \quad t > 0), \\ u(0,t) &= u(L,t) = 0 && (t > 0), \\ u(x,0) &= f(x) && (0 < x < L). \end{aligned}$$

We seek separated solutions of the form $u(x,t) = X(x)T(t)$. In this case

$$\left. \begin{aligned} u_t &= XT' \\ u_{xx} &= X''T \end{aligned} \right\} \Rightarrow \quad XT' = c^2 X'' T \quad \Rightarrow \quad \frac{X''}{X} = \frac{T'}{c^2 T} = k.$$

Together with the boundary conditions we obtain the system

$$X'' - kX = 0, \quad X(0) = X(L) = 0,$$

$$T' - c^2 kT = 0.$$

Already know: up to constant multiples, the only solutions to the BVP in X are

$$k = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2,$$
$$X = X_n = \sin(\mu_n x) = \sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}.$$

Therefore T must satisfy

$$T' - c^2 kT = T' + \underbrace{\left(\frac{cn\pi}{L}\right)^2}_{\lambda_n} T = 0$$

$$T' = -\lambda_n^2 T \quad \Rightarrow \quad T = T_n = b_n e^{-\lambda_n^2 t}.$$

We thus have the *normal modes* of the heat equation:

$$u_n(x,t) = X_n(x) T_n(t) = b_n e^{-\lambda_n^2 t} \sin(\mu_n x), \quad n \in \mathbb{N}.$$

Superposition and initial condition

Applying the principle of superposition gives the general solution

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}b_n e^{-\lambda_n^2 t}\sin(\mu_n x).$$

If we now impose our initial condition we find that

$$f(x)=u(x,0)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right),$$

which is the sine series expansion of $f(x)$. Hence

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Remarks

- As before, if the sine series of $f(x)$ is already known, solution can be built by simply including exponential factors.
- One can show that this is the *only* solution to the heat equation with the given initial condition.
- Because of the decaying exponential factors:
  - The normal modes tend to zero (exponentially) as $t \to \infty$.
  - Overall, $u(x,t) \to 0$ (exponentially) *uniformly in* x as $t \to \infty$.
  - As c increases, $u(x,t) \to 0$ more rapidly.

This agrees with intuition.

Example

Solve the heat problem

$$\begin{aligned} u_t &= 3u_{xx} & (0 < x < 2, \quad t > 0), \\ u(0,t) &= u(2,t) = 0 & (t > 0), \\ u(x,0) &= 50 & (0 < x < 2). \end{aligned}$$

We have $c = \sqrt{3}$, $L = 2$ and, by exercise 2.3.1 (with $p = L = 2$)

$$f(x) = 50 = \frac{200}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin\left(\frac{(2k+1)\pi x}{2}\right).$$

Since $\lambda_{2k+1} = \dfrac{c(2k+1)\pi}{L} = \dfrac{\sqrt{3}(2k+1)\pi}{2}$, we obtain

$$u(x,t) = \frac{200}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} e^{-3(2k+1)^2\pi^2 t/4} \sin\left(\frac{(2k+1)\pi x}{2}\right).$$

Solving the Heat Equation

Case 2a: steady state solutions

Definition: We say that $u(x,t)$ is a *steady state solution* if $u_t \equiv 0$ (i.e. u is time-independent).

If $u(x,t)$ is a steady state solution to the heat equation then

$$u_t \equiv 0 \;\Rightarrow\; c^2 u_{xx} = u_t = 0 \;\Rightarrow\; u_{xx} = 0 \;\Rightarrow\; u = Ax + B.$$

Steady state solutions can help us deal with inhomogeneous Dirichlet boundary conditions. Note that

$$\left.\begin{aligned} u(0,t) &= T_1 \\ u(L,t) &= T_2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} B &= T_1 \\ AL + B &= T_2 \end{aligned}\right\} \Rightarrow u = \left(\frac{T_2 - T_1}{L}\right) x + T_1.$$

Solving the Heat Equation

Case 2b: inhomogeneous Dirichlet boundary conditions

Now consider the heat problem

$$\begin{aligned} &u_t = c^2 u_{xx} && (0 < x < L, \ \ t > 0), \\ &u(0,t) = T_1, \ \ u(L,t) = T_2 && (t > 0), \\ &u(x,0) = f(x) && (0 < x < L). \end{aligned}$$

Step 1: Let u_1 denote the steady state solution from above:

$$u_1 = \left(\frac{T_2 - T_1}{L}\right) x + T_1.$$

Step 2: Let $u_2 = u - u_1$.

Remark: By superposition, u_2 still solves the heat equation.

The boundary and initial conditions satisfied by u_2 are

$$\begin{aligned} u_2(0,t) &= u(0,t) - u_1(0) = T_1 - T_1 = 0, \\ u_2(L,t) &= u(L,t) - u_1(L) = T_2 - T_2 = 0, \\ u_2(x,0) &= f(x) - u_1(x). \end{aligned}$$

Step 3: Solve the heat equation with homogeneous Dirichlet boundary conditions and initial conditions above. This yields u_2.

Step 4: Assemble $u(x,t) = u_1(x) + u_2(x,t)$.

Remark: According to our earlier work, $\lim_{t\to\infty} u_2(x,t) = 0$.

- We call $u_2(x,t)$ the *transient* portion of the solution.
- We have $u(x,t) \to u_1(x)$ as $t \to \infty$, i.e. the solution tends to the steady state.

Example

Solve the heat problem.

$$\begin{aligned} &u_t = 3u_{xx} && (0 < x < 2,\ \ t > 0),\\ &u(0,t) = 100,\quad u(2,t) = 0 && (t > 0),\\ &u(x,0) = 50 && (0 < x < 2). \end{aligned}$$

We have $c = \sqrt{3}$, $L = 2$, $T_1 = 100$, $T_2 = 0$ and $f(x) = 50$. The steady state solution is

$$u_1 = \left(\frac{0 - 100}{2}\right) x + 100 = 100 - 50x.$$

The corresponding homogeneous problem for u_2 is thus

$$\begin{aligned} &u_t = 3u_{xx} && (0 < x < 2,\ \ t > 0),\\ &u(0,t) = u(2,t) = 0 && (t > 0),\\ &u(x,0) = 50 - (100 - 50x) \quad = 50(x - 1) && (0 < x < 2). \end{aligned}$$

According to exercise 2.3.7 (with $p = L = 2$), the sine series for $50(x-1)$ is

$$\frac{-100}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin\left(\frac{2k\pi x}{2}\right),$$

i.e. only *even* modes occur. Since $\lambda_{2k} = \dfrac{c2k\pi}{L} = \sqrt{3}k\pi$,

$$u_2(x,t) = \frac{-100}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}e^{-3k^2\pi^2 t}\sin(k\pi x).$$

Hence

$$u(x,t) = u_1(x) + u_2(x,t) = 100 - 50x - \frac{100}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}e^{-3k^2\pi^2 t}\sin(k\pi x).$$