The One-Dimensional Heat Equation: Neumann and Robin boundary conditions

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Partial Differential Equations
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A heat problem with Neumann boundary conditions

Goal: Solve the following heat problem:

$$
\begin{aligned}
u_t &= c^2 u_{xx} && (0 < x < L\ ,\ 0 < t),\\
u_x(0,t) &= u_x(L,t) = 0 && (0 < t),\\
u(x,0) &= f(x) && (0 < x < L).
\end{aligned}
$$

This models the heat flow in a wire of length L with given initial temperature distribution and *insulated ends*.

As before, assuming $u(x,t) = X(x)T(t)$ yields the system

$$
\begin{aligned}
&X'' - kX = 0,\ \ X'(0) = X'(L) = 0,\\
&T' - c^2 kT = 0.
\end{aligned}
$$

Note that the boundary conditions on X *are not the same* as in the Dirichlet condition case.

Solving for X

Case 1: $k = \mu^2 > 0$. We need to solve $X'' - \mu^2 X = 0$. The characteristic equation is

$$r^2 - \mu^2 = 0 \quad \Rightarrow \quad r = \pm\mu,$$

which gives the general solution $X = c_1 e^{\mu x} + c_2 e^{-\mu x}$. The boundary conditions tell us that

$$0 = X'(0) = \mu c_1 - \mu c_2, \;\; 0 = X'(L) = \mu c_1 e^{\mu L} - \mu c_2 e^{-\mu L},$$

or in matrix form

$$\begin{pmatrix} \mu & -\mu \\ \mu e^{\mu L} & -\mu e^{-\mu L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the determinant is $\mu^2(e^{\mu L} - e^{-\mu L}) \neq 0$, we must have $c_1 = c_2 = 0$, and so $X \equiv 0$.

Case 2: $k = 0$. We need to solve $X'' = 0$. Integrating twice gives

$$X = c_1 x + c_2.$$

The boundary conditions give $0 = X'(0) = X'(L) = c_1$. Taking $c_2 = 1$ we get the solution

$$X = X_0 = 1.$$

Case 3: $k = -\mu^2 < 0$. We need to solve $X'' + \mu^2 X = 0$. The characteristic equation is

$$r^2 + \mu^2 = 0 \quad \Rightarrow \quad r = \pm i\mu,$$

which gives the general solution $X = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

The boundary conditions yield

$$\begin{aligned} 0 = X'(0) &= -\mu c_1 \sin 0 + \mu c_2 \cos 0 = \mu c_2 \quad \Rightarrow \quad c_2 = 0, \\ 0 = X'(L) &= -\mu c_1 \sin(\mu L) + \mu c_2 \cos(\mu L) = -\mu c_1 \sin(\mu L). \end{aligned}$$

In order to have $X \not\equiv 0$, this shows that we need

$$\sin(\mu L) = 0 \quad \Rightarrow \quad \mu L = n\pi \quad \Rightarrow \quad \mu = \mu_n = \frac{n\pi}{L} \quad (n \in \mathbb{Z}).$$

Taking $c_1 = 1$ we obtain

$$X = X_n = \cos(\mu_n x) \quad (n \in \mathbb{N}).$$

Remarks:

- We only need $n > 0$, since cosine is an even function.
- When $n = 0$ we get $X_0 = \cos 0 = 1$, which agrees with the $k = 0$ result.

Normal modes and superposition

As before, for $k = -\mu_n^2$, we obtain $T = T_n = a_n e^{-\lambda_n^2 t}$.

We therefore have the normal modes

$$u_n(x,t) = X_n(x)T_n(t) = a_n e^{-\lambda_n^2 t}\cos(\mu_n x) \quad (n \in \mathbb{N}_0),$$

where $\mu_n = n\pi/L$ and $\lambda_n = c\mu_n$.

The principle of superposition now gives the general solution

$$u(x,t) = u_0 + \sum_{n=1}^{\infty} u_n = a_0 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t}\cos(\mu_n x)$$

to the heat equation with (homogeneous) Neumann boundary conditions.

Initial conditions

If we now impose our initial condition we find that

$$f(x) = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

which is simply the $2L$-periodic cosine expansion of $f(x)$. Hence

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad (n \in \mathbb{N}).$$

Remarks:

- As before, if the cosine series of $f(x)$ is already known, $u(x,t)$ can be built by simply including exponential factors.
- Because of the exponential factors, $\lim_{t\to\infty} u(x,t) = a_0$, which is the *average initial temperature*.

Example

Solve the following heat problem:

$$\begin{aligned} u_t &= \frac{1}{4} u_{xx}, && 0 < x < 1 \ , \ 0 < t, \\ u_x(0,t) &= u_x(1,t) = 0, && 0 < t, \\ u(x,0) &= 100x(1-x), && 0 < x < 1. \end{aligned}$$

We have $c = 1/2$, $L = 1$ and $f(x) = 100x(1-x)$. Therefore

$$a_0 = \int_0^1 100x(1-x)\,dx = \frac{50}{3}$$

$$a_n = 2\int_0^1 100x(1-x)\cos n\pi x\,dx = \frac{-200(1+(-1)^n)}{n^2\pi^2}, \ \ n \geq 1.$$

Example 1

Since $\lambda_n = cn\pi/L = n\pi/2$, plugging everything into the general solution we get

$$u(x,t) = \frac{50}{3} - \frac{200}{\pi^2}\sum_{n=1}^{\infty}\frac{(1+(-1)^n)}{n^2}e^{-n^2\pi^2 t/4}\cos n\pi x.$$

As in the case of Dirichlet boundary conditions, the exponential terms decay rapidly with t. We therefore have

$$\lim_{t\to\infty} u(x,t) = \frac{50}{3}.$$

A heat problem with Robin boundary conditions

Goal: Solve the heat problem

$$\begin{aligned} u_t &= c^2 u_{xx} && (0 < x < L,\ 0 < t), \\ u(0,t) &= 0 && (0 < t), \\ u_x(L,t) &= -\kappa u(L,t) && (0 < t), \\ u(x,0) &= f(x) && (0 < x < L). \end{aligned} \tag{1}$$

Remarks:

- The condition (1) is linear and homogeneous:

$$\kappa u(L,t) + u_x(L,t) = 0$$

  Recall that this is called a *Robin condition*.

- We take $\kappa > 0$. This means that the bar radiates heat to its surroundings at a rate proportional to its current temperature.

Separation of variables

As before, the assumption that $u(x,t) = X(x)T(t)$ leads to the ODEs

$$X'' - kX = 0, \quad T' - c^2 kT = 0,$$

and the boundary conditions imply

$$X(0) = 0, \quad X'(L) = -\kappa X(L).$$

Also as before, we solve for X first.

Case 1: $k = 0$. As above, solving $X'' = 0$ gives $X = c_1 x + c_2$. The boundary conditions become

$$0 = X(0) = c_2, \quad c_1 = X'(L) = -\kappa X(L) = -\kappa(c_1 L + c_2)$$
$$\Rightarrow \quad c_1(1 + \kappa L) = 0 \quad \Rightarrow \quad c_1 = 0.$$

Hence, $X \equiv 0$ in this case.

Case 1: $k = \mu^2 > 0$. Again we have $X'' - \mu^2 X = 0$ and

$$X = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$

The boundary conditions become

$$0 = c_1 + c_2, \quad \mu(c_1 e^{\mu L} - c_2 e^{-\mu L}) = -\kappa(c_1 e^{\mu L} + c_2 e^{-\mu L}),$$

or in matrix form

$$\begin{pmatrix} 1 & 1 \\ (\kappa + \mu)e^{\mu L} & (\kappa - \mu)e^{-\mu L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant is

$$(\kappa-\mu)e^{-\mu L}-(\kappa+\mu)e^{\mu L} = -\left(\kappa(e^{\mu L} - e^{-\mu L}) + \mu(e^{\mu L} + e^{-\mu L})\right) < 0,$$

so that $c_1 = c_2 = 0$ and $X \equiv 0$.

Case 3: $k = -\mu^2 < 0$. From $X'' + \mu^2 X = 0$ we find

$$X = c_1 \cos(\mu x) + c_2 \sin(\mu x)$$

and from the boundary conditions we have

$$0 = c_1, \quad \mu(-c_1 \sin(\mu L) + c_2 \cos(\mu L)) = -\kappa(c_1 \cos(\mu L) + c_2 \sin(\mu L))$$
$$\Rightarrow \quad c_2 \left(\mu \cos(\mu L) + \kappa \sin(\mu L)\right) = 0.$$

So that $X \not\equiv 0$, we must have

$$\mu \cos(\mu L) + \kappa \sin(\mu L) = 0 \ \Rightarrow \ \tan(\mu L) = -\frac{\mu}{\kappa}.$$

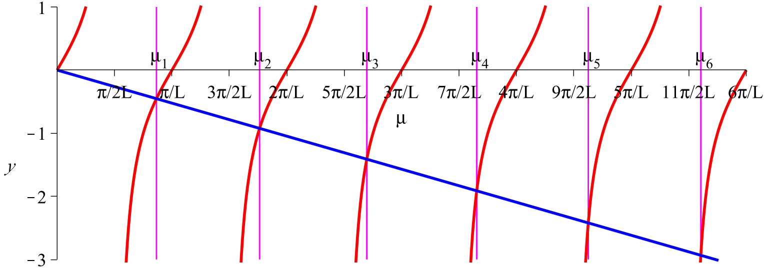
This equation has an infinite sequence of positive solutions

$$0 < \mu_1 < \mu_2 < \mu_3 < \cdots$$

and we obtain $X = X_n = \sin(\mu_n x)$ for $n \in \mathbb{N}$.

The solutions of $\tan(\mu L) = -\mu/\kappa$

The figure below shows the curves $y = \tan(\mu L)$ (in red) and $y = -\mu/\kappa$ (in blue).



The μ-coordinates of their intersections (in pink) are the values μ_1, μ_2, μ_3, . . .

Remarks: From the diagram we see that:

- For each n, $(2n-1)\pi/2L < \mu_n < n\pi/L$.
- As $n \to \infty$, $\mu_n \to (2n-1)\pi/2L$.
- Smaller values of κ and L tend to accelerate this convergence.

Normal modes: As in the earlier situations, for each $n \in \mathbb{N}$ we have the corresponding

$$T = T_n = c_n e^{-\lambda_n^2 t}, \ \lambda_n = c\mu_n$$

which gives the *normal mode*

$$u_n(x,t) = X_n(x)T_n(t) = c_n e^{-\lambda_n^2 t}\sin(\mu_n x).$$

Superposition

Superposition of normal modes gives the general solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n^2 t} \sin(\mu_n x).$$

Imposing the initial condition gives us

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin(\mu_n x).$$

This is a *generalized Fourier sine series* for $f(x)$. It is *different* from the ordinary sine series for $f(x)$ since

μ_n is not a multiple of π/L.

Generalized Fourier coefficients

To compute the *generalized Fourier coefficients* c_n we will use:

Theorem

The functions

$$X_1(x) = \sin(\mu_1 x), X_2(x) = \sin(\mu_2 x), X_3(x) = \sin(\mu_3 x), \dots$$

form a complete orthogonal set on $[0, L]$*.*

- *Complete* means that all "sufficiently nice" functions can be represented via generalized Fourier series.
- Recall that $f(x)$ and $g(x)$ are orthogonal on $[0, L]$ provided

$$\langle f, g \rangle = \int_0^L f(x)g(x)\,dx = 0.$$

"Extracting" the generalized Fourier coefficients

If $f(x) = \sum_{n=1}^{\infty} c_n \sin(\mu_n x) = \sum_{n=1}^{\infty} c_n X_n(x)$ and $m \in \mathbb{N}$ we have

$$\begin{aligned} \langle f, X_m \rangle = \left\langle \sum_{n=1}^{\infty} c_n X_n, X_m \right\rangle &= \sum_{n=1}^{\infty} c_n \langle X_n, X_m \rangle \\ &= c_m \langle X_m, X_m \rangle \end{aligned}$$

since $\langle X_n, X_m \rangle = 0$ for $n \neq m$. It follows that the generalized Fourier coefficients are given by

$$c_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle} = \frac{\displaystyle\int_0^L f(x) \sin(\mu_n x)\, dx}{\displaystyle\int_0^L \sin^2(\mu_n x)\, dx}.$$

Conclusion

Theorem

The solution to the heat problem with boundary and initial conditions

$$\begin{aligned} &u(0,t)=0,\ \ u_x(L,t)=-\kappa u(L,t) && (0<t),\\ &u(x,0)=f(x) && (0<x<L) \end{aligned}$$

is given by $u(x,t)=\sum_{n=1}^{\infty} c_n e^{-\lambda_n^2 t}\sin(\mu_n x)$, *where* μ_n *is the nth positive solution to* $\tan(\mu L)=\frac{-\mu}{\kappa}$, $\lambda_n=c\mu_n$, *and*

$$c_n=\frac{\int_0^L f(x)\sin(\mu_n x)\,dx}{\int_0^L \sin^2(\mu_n x)\,dx}.$$

Remarks:

- For any given $f(x)$ these integrals can be computed explicitly in terms of μ_n.
- The values of μ_n, however, must typically be found via numerical methods.

Example

Solve the following heat problem:

$$\begin{aligned} u_t &= \frac{1}{25} u_{xx} && (0 < x < 3,\ 0 < t), \\ u(0,t) &= 0, \quad u_x(3,t) = -\frac{1}{2} u(3,t) && (0 < t), \\ u(x,0) &= 100\left(1 - \frac{x}{3}\right) && (0 < x < 3). \end{aligned}$$

We have $c = 1/5$, $L = 3$, $\kappa = 1/2$ and $f(x) = 100(1 - x/3)$.

The integrals defining the Fourier coefficients are

$$100\int_0^3 \left(1-\frac{x}{3}\right)\sin(\mu_n x)\,dx = \frac{100(3\mu_n - \sin(3\mu_n))}{3\mu_n^2}$$

and

$$\int_0^3 \sin^2(\mu_n x)\,dx = \frac{3}{2} + \cos^2(3\mu_n).$$

Hence

$$c_n = \frac{200(3\mu_n - \sin(3\mu_n))}{3\mu_n^2\,(3+2\cos^2(3\mu_n))}.$$

We therefore have

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200(3\mu_n - \sin(3\mu_n))}{3\mu_n^2\,(3+2\cos^2(3\mu_n))} e^{-\mu_n^2 t/25}\sin(\mu_n x),$$

where μ_n is the nth positive solution to $\tan(3\mu) = -2\mu$.

Remarks:

- In order to use this solution for numerical approximation or visualization, we must compute the values μ_n.
- This can be done numerically in Maple, using the `fsolve` command. Specifically, μ_n can be computed via the input

```
fsolve(tan(m*L)=-m/k,m=(2*n-1)*Pi/(2*L)..n*Pi/L);
```

  where `L` and `k` have been assigned the values of L and κ, respectively.
- These values can be computed and stored in an `Array` structure, or one can define μ_n as a function using the `->` operator.

Here are approximations to the first 5 values of μ_n and c_n in the preceding example.

n	μ_n	c_n
1	0.7249	47.0449
2	1.6679	45.1413
3	2.6795	21.3586
4	3.7098	19.3403
5	4.7474	12.9674

Therefore

$$\begin{aligned} u(x,t) &= 47.0449e^{-0.0210t}\sin(0.7249x) + 45.1413e^{-0.1113t}\sin(1.6679x) \\ &\quad + 21.3586e^{-0.2872t}\sin(2.6795x) + 19.3403e^{-0.5505t}\sin(3.7098x) \\ &\quad + 12.9674e^{-0.9015t}\sin(4.7474x) + \cdots \end{aligned}$$