

# Integral formulas for Fourier coefficients

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**Recall:** Relative to the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) dx$$

the functions

$$1, \cos(x), \cos(2x), \cos(3x), \dots, \sin(x), \sin(2x), \sin(3x), \dots$$

satisfy the *orthogonality relations*

$$\langle \cos(mx), \sin(nx) \rangle = 0,$$

$$\langle \cos(mx), \cos(nx) \rangle = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n \neq 0, \\ 2\pi & \text{if } m = n = 0, \end{cases}$$

$$\langle \sin(mx), \sin(nx) \rangle = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$$

By the linearity of the inner product, if

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)),$$

then

$$\begin{aligned} \langle f(x), \cos(mx) \rangle &= a_0 \langle 1, \cos(mx) \rangle + \sum_{n=1}^{\infty} (a_n \langle \cos(nx), \cos(mx) \rangle \\ &\quad + b_n \underbrace{\langle \sin(nx), \cos(mx) \rangle}_{=0}) \\ &= a_0 \underbrace{\langle 1, \cos(mx) \rangle}_{=0 \text{ unless } m=0} + \sum_{n=1}^{\infty} a_n \underbrace{\langle \cos(nx), \cos(mx) \rangle}_{=0 \text{ unless } m=n} \\ &= \begin{cases} 2\pi a_0 & \text{if } m = 0, \\ \pi a_m & \text{if } m \neq 0. \end{cases} \end{aligned}$$

Likewise, one can show that

$$\langle f(x), \sin(mx) \rangle = \pi b_m.$$

Solving for the coefficients gives:

### Theorem (Euler's Formulas)

*If  $f$  is  $2\pi$ -periodic and piecewise smooth, then its Fourier coefficients are given by*

$$a_0 = \frac{\langle f(x), 1 \rangle}{2\pi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx,$$

$$a_n = \frac{\langle f(x), \cos(nx) \rangle}{\pi} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad (n \neq 0),$$

$$b_n = \frac{\langle f(x), \sin(nx) \rangle}{\pi} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx.$$

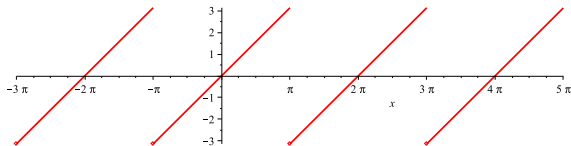
# Remarks

- Technically we should have used  $\frac{f(x+) + f(x-)}{2}$ . However, the integrals cannot distinguish between this and  $f(x)$ .
- Because all the functions in question are  $2\pi$ -periodic, we can integrate over *any* convenient interval of length  $2\pi$ .
- If  $f(x)$  is an odd function, so is  $f(x) \cos(nx)$ , and so  $a_n = 0$  for all  $n \geq 0$ .
- If  $f(x)$  is an even function, then  $f(x) \sin(nx)$  is odd, and so  $b_n = 0$  for all  $n \geq 1$ .

### Example

Find the Fourier series for the  $2\pi$ -periodic function that satisfies  $f(x) = x$  for  $-\pi < x \leq \pi$ .

The graph of  $f$  (a sawtooth wave):



Because  $f$  is odd, we know

$$a_n = 0 \quad (n \geq 0).$$

According to Euler's formula:

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx = 0 \\
 &= \frac{1}{\pi} \left( \frac{-x \cos(nx)}{n} + \frac{\sin(nx)}{n^2} \Big|_{-\pi}^{\pi} \right) \\
 &= \frac{1}{\pi} \left( \frac{-\pi \cos(n\pi)}{n} - \frac{\pi \cos(-n\pi)}{n} \right) \\
 &= \frac{-2 \cos(n\pi)}{n} = \frac{(-1)^{n+1} 2}{n}.
 \end{aligned}$$

Therefore, the Fourier series of  $f$  is

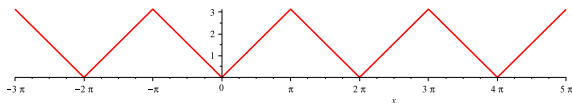
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2}{n} \sin(nx) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(nx)}{n}.$$

**Remark:** Except where it is discontinuous, this series equals  $f(x)$ .

### Example

Find the Fourier series of the  $2\pi$ -periodic function satisfying  $f(x) = |x|$  for  $-\pi \leq x < \pi$ .

The graph of  $f$  (a *triangular wave*):



This time, since  $f$  is even,

$$b_n = 0 \quad (n \geq 1).$$



By Euler's formula we have

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx = \frac{1}{\pi} \int_0^{\pi} x dx = \frac{1}{\pi} \left. \frac{x^2}{2} \right|_0^{\pi} = \frac{\pi}{2}$$

and for  $n \geq 1$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{|x| \cos(nx)}_{\text{even}} dx = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx \\ &= \frac{2}{\pi} \left( \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right) \Big|_0^{\pi} = \frac{2}{\pi} \left( \frac{\cos(n\pi)}{n^2} - \frac{1}{n^2} \right) \\ &= \frac{2}{\pi n^2} ((-1)^n - 1) = \begin{cases} \frac{-4}{\pi n^2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

In the Fourier series we may therefore omit the terms in which  $n$  is even, and assume that  $n = 2k + 1$ ,  $k \geq 0$ :

$$\begin{aligned} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) &= \frac{\pi}{2} + \sum_{k=0}^{\infty} \frac{-4}{\pi(2k+1)^2} \cos((2k+1)x) \\ &= \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}. \end{aligned}$$

### Remarks:

- Since  $k$  is simply an index of summation, we are free to replace it with  $n$  again, yielding

$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos((2n+1)x)}{(2n+1)^2}.$$

- Because  $f(x)$  is continuous everywhere, this equals  $f(x)$  at all points.

### Example

Use the result of the previous exercise to show that

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

If we set  $x = 0$  in the previous example, we get

$$0 = f(0) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(0)}{(2n+1)^2} = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

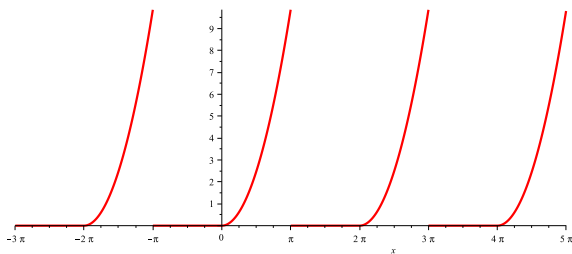
Solving for the series gives the result.

**Remark:** In Calculus II you learned that this series converges, but were unable to obtain its exact value.

### Example

Find the Fourier series of the  $2\pi$ -periodic function satisfying  $f(x) = 0$  for  $-\pi \leq x < 0$  and  $f(x) = x^2$  for  $0 \leq x < \pi$ .

The graph of  $f$ :



Because  $f$  is neither even nor odd, we must compute all of its Fourier coefficients directly.

Since  $f(x) = 0$  for  $-\pi \leq x < 0$ , Euler's formulas become

$$a_0 = \frac{1}{2\pi} \int_0^\pi x^2 dx = \frac{1}{2\pi} \left( \frac{x^3}{3} \Big|_0^\pi \right) = \frac{\pi^2}{6}$$

and for  $n \geq 1$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^\pi x^2 \cos(nx) dx = \frac{1}{\pi} \left( \frac{x^2 \sin(nx)}{n} + \frac{2x \cos(nx)}{n^2} - \frac{2 \sin(nx)}{n^3} \Big|_0^\pi \right) \\ &= \frac{1}{\pi} \cdot \frac{2\pi \cos(n\pi)}{n^2} = \frac{2(-1)^n}{n^2}, \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^\pi x^2 \sin(nx) dx = \frac{1}{\pi} \left( -\frac{x^2 \cos(nx)}{n} + \frac{2x \sin(nx)}{n^2} + \frac{2 \cos(nx)}{n^3} \Big|_0^\pi \right) \\ &= \frac{1}{\pi} \left( -\frac{\pi^2 \cos(n\pi)}{n} + \frac{2 \cos(n\pi)}{n^3} - \frac{2}{n^3} \right) = \frac{(-1)^{n+1} \pi}{n} + \frac{2((-1)^n - 1)}{\pi n^3}. \end{aligned}$$

Therefore the Fourier series of  $f$  is

$$\frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left( \frac{2(-1)^n}{n^2} \cos(nx) + \left( \frac{(-1)^{n+1}\pi}{n} + \frac{2((-1)^n - 1)}{\pi n^3} \right) \sin(nx) \right).$$

This will agree with  $f(x)$  everywhere it's continuous.

# Convergence of Fourier series

Given a Fourier series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \quad (1)$$

let its  $N$ th *partial sum* be

$$s_N(x) = a_0 + \sum_{n=1}^N (a_n \cos(nx) + b_n \sin(nx)). \quad (2)$$

According to the definition of an infinite series, the Fourier series (1) is equal to

$$\lim_{N \rightarrow \infty} s_N(x).$$

According to the definition of the limit:

- We can *approximate* the (infinite) Fourier series (1) by the (finite) partial sums (2).
- The approximation of (1) by (2) improves (indefinitely) as we increase  $N$ .

Because  $s_N(x)$  is a *finite* sum, we can use a computer to graph it.

In this way, we can visualize the convergence of a Fourier series.

Let's look at some examples...



These examples illustrate the following results. In both,  $f(x)$  is  $2\pi$ -periodic and piecewise smooth.

### Theorem (Uniform convergence of Fourier series)

*If  $f(x)$  is continuous everywhere, then the partial sums  $s_N(x)$  of its Fourier series converge uniformly to  $f(x)$  as  $N \rightarrow \infty$ .*

*That is, by choosing  $N$  large enough we can make  $s_N(x)$  arbitrarily close to  $f(x)$  for all  $x$  simultaneously.*

### Theorem (Wilbraham-Gibbs phenomenon)

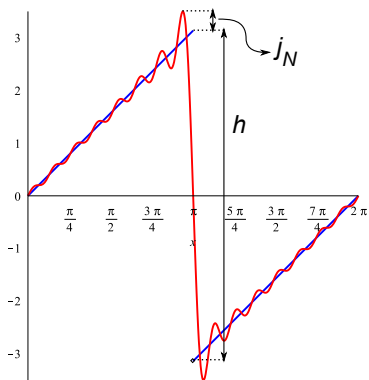
*If  $f(x)$  has a jump discontinuity at  $x = c$ , then the partial sums  $s_N(x)$  of its Fourier series always "overshoot"  $f(x)$  near  $x = c$ .*

*More precisely, as  $N \rightarrow \infty$ , the the ratio between the peak of the overshoot and the height of the jump tends to*

$$\frac{1}{\pi} \int_0^{\pi} \frac{\sin t}{t} dt - \frac{1}{2} = 0.08948 \dots \quad (\text{about } 9\% \text{ of the jump}).$$

# The Wilbraham-Gibbs phenomenon

A function  $f(x)$  (in blue) with a jump discontinuity and a partial sum  $s_N(x)$  (in red) of its Fourier series:



$$\lim_{N \rightarrow \infty} \frac{j_N}{h} = 0.08948 \dots$$

## General Fourier series

If  $f(x)$  is  $2p$ -periodic and piecewise smooth, then  $\hat{f}(x) = f(px/\pi)$  has period  $\frac{2p}{p/\pi} = 2\pi$ , and is also piecewise smooth.

It follows that  $\hat{f}(x)$  has a Fourier series:

$$\frac{\hat{f}(x+) + \hat{f}(x-)}{2} = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)).$$

Since  $f(x) = \hat{f}(\pi x/p)$ , we find that  $f$  also has a Fourier series:

$$\frac{f(x+) + f(x-)}{2} = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right).$$

We can use Euler's formulas to find  $a_n$  and  $b_n$ . For example

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{px}{\pi}\right) dx = \frac{1}{2p} \int_{-p}^p f(t) dt,$$

where in the final equality we used the substitution  $t = px/\pi$ .

In the same way one can show that for  $n \geq 1$

$$a_n = \frac{1}{p} \int_{-p}^p f(t) \cos\left(\frac{n\pi t}{p}\right) dt,$$
$$b_n = \frac{1}{p} \int_{-p}^p f(t) \sin\left(\frac{n\pi t}{p}\right) dt.$$

Since  $t$  is simply a “dummy” variable of integration, we may replace it with  $x$  in each case.

## Remarks on general Fourier series

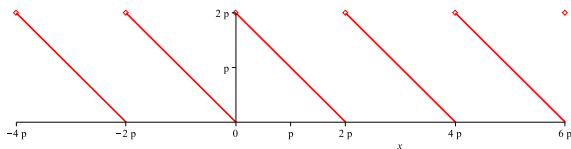
Everything we've done with  $2\pi$ -periodic Fourier series continues to hold in this case, with  $p$  replacing  $\pi$ :

- We can compute general Fourier coefficients by integrating over any “convenient” interval of length  $2p$ .
- If  $p$  is left unspecified, then the formulae for  $a_n$  and  $b_n$  may involve  $p$ .
- If  $f(x)$  is even, then  $b_n = 0$  for all  $n$ .
- If  $f(x)$  is odd, then  $a_n = 0$  for all  $n$ .
- We still have the uniform convergence theorem and Wilbraham-Gibbs phenomenon.

### Example

Find the Fourier series of the  $2p$ -periodic function that satisfies  $f(x) = 2p - x$  for  $0 \leq x < 2p$ .

The graph of  $f(x)$ :



We will use Euler's formulas over the interval  $[0, 2p]$  to simplify our calculations.

We have

$$a_0 = \frac{1}{2p} \int_0^{2p} 2p - x \, dx = \frac{1}{2p} \left( 2px - \frac{x^2}{2} \Big|_0^{2p} \right) = p$$

and for  $n \geq 1$

$$\begin{aligned} a_n &= \frac{1}{p} \int_0^{2p} (2p - x) \cos\left(\frac{n\pi x}{p}\right) dx \\ &= \frac{1}{p} \left( \frac{p(2p - x) \sin\left(\frac{n\pi x}{p}\right)}{n\pi} - \frac{p^2 \cos\left(\frac{n\pi x}{p}\right)}{n^2 \pi^2} \Big|_0^{2p} \right) \\ &= \frac{1}{p} \left( -\frac{p^2 \cos(2n\pi)}{n^2 \pi^2} + \frac{p^2}{n^2 \pi^2} \right) = 0, \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{p} \int_0^{2p} (2p - x) \sin\left(\frac{n\pi x}{p}\right) dx \\ &= \frac{1}{p} \left( \frac{-p(2p - x) \cos\left(\frac{n\pi x}{p}\right)}{n\pi} - \frac{p^2 \sin\left(\frac{n\pi x}{p}\right)}{n^2 \pi^2} \Big|_0^{2p} \right) \\ &= \frac{1}{p} \left( \frac{2p^2}{n\pi} \right) = \frac{2p}{n\pi}. \end{aligned}$$

So the Fourier series of  $f$  is

$$p + \sum_{n=1}^{\infty} \frac{2p}{n\pi} \sin\left(\frac{n\pi x}{p}\right) = p + \frac{2p}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{p}\right).$$

**Remark:** This series is equal to  $f(x)$  everywhere it is continuous.