The Dirichlet Problem on a Rectangle

R. C. Daileda

Trinity University

Partial Differential Equations
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Recall: A *steady state* solution to a (time-dependent) PDE satisfies $u_t \equiv 0$.

- Steady state solutions of the 1-D heat equation $u_t = c^2 u_{xx}$ satisfy

$$u_{xx} = 0,$$

  i.e. are simply straight lines.

- Steady state solutions of the 2-D heat equation $u_t = c^2 \nabla^2 u$ satisfy

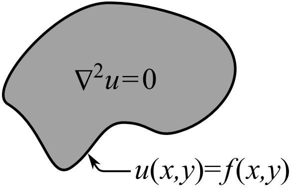
$$\nabla^2 u = u_{xx} + u_{yy} = 0 \quad \text{(Laplace's equation)},$$

  and are called *harmonic functions.*

Dirichlet problems

Definition: The *Dirichlet problem* on a region $R \subseteq \mathbb{R}^2$ is the boundary value problem

$$\nabla^2 u = 0 \ \text{ inside } R$$
$$u(x,y) = f(x,y) \ \text{ on } \partial R.$$



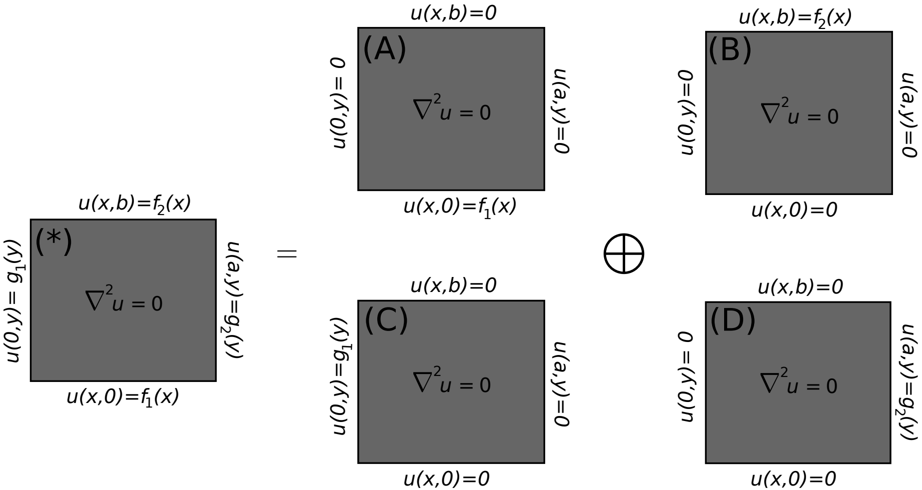
For simplicity we will assume that:

- The region is rectangular: $R = [0,a] \times [0,b]$.
- The boundary conditions are given on each edge separately.

$$u(x,0) = f_1(x), \qquad u(x,b) = f_2(x), \qquad 0 < x < a,$$
$$u(0,y) = g_1(y), \qquad u(a,y) = g_2(y), \qquad 0 < y < b.$$

Solving the Dirichlet problem on a rectangle

Strategy: Reduce to four simpler problems and use superposition.



Remarks:

- If u_1, u_2, u_3 and u_4 solve the Dirichlet problems (A), (B), (C) and (D) (respectively), then the general solution to $(*)$ is

$$u = u_1 + u_2 + u_3 + u_4.$$

- The boundary conditions in (A) - (D) are all homogeneous, with the exception of a single edge.

- Problems with inhomogeneous Neumann or Robin boundary conditions (or combinations thereof) can be reduced in a similar manner.

Solution of the Dirichlet problem on a rectangle
Case B

Goal: Solve the boundary value problem

$$\begin{aligned}
\nabla^2 u &= 0, && 0 < x < a,\ 0 < y < b,\\
u(x,0) = 0,\ u(x,b) &= f_2(x), && 0 < x < a,\\
u(0,y) = u(a,y) &= 0, && 0 < y < b.
\end{aligned}$$

Setting $u(x,y) = X(x)Y(y)$ leads to

$$\begin{aligned}
X'' + kX &= 0, \qquad & Y'' - kY &= 0,\\
X(0) = X(a) &= 0, \qquad & Y(0) &= 0.
\end{aligned}$$

We know the nontrivial solutions for X are given by

$$X(x) = X_n(x) = \sin(\mu_n x),\ \ \mu_n = \frac{n\pi}{a},\ \ k = \mu_n^2 \quad (n \in \mathbb{N}).$$

Interlude

The hyperbolic trigonometric functions

The *hyperbolic cosine and sine functions* are

$$\cosh y = \frac{e^y + e^{-y}}{2}, \quad \sinh y = \frac{e^y - e^{-y}}{2}.$$

They satisfy the following identities:

$$\cosh^2 y - \sinh^2 y = 1,$$
$$\frac{d}{dy}\cosh y = \sinh y, \quad \frac{d}{dy}\sinh y = \cosh y.$$

It follows that the general solution to the ODE $Y'' - \mu^2 Y = 0$ is

$$Y = A\cosh(\mu y) + B\sinh(\mu y).$$

Using $\mu = \mu_n$ and $Y(0) = 0$, we find

$$\begin{aligned} Y(y) = Y_n(y) &= A_n \cosh(\mu_n y) + B_n \sinh(\mu_n y) \\ 0 = Y_n(0) &= A_n \cosh 0 + B_n \sinh 0 = A_n. \end{aligned}$$

This yields the separated solutions

$$u_n(x, y) = X_n(x) Y_n(y) = B_n \sin(\mu_n x) \sinh(\mu_n y),$$

and superposition gives the general solution

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin(\mu_n x) \sinh(\mu_n y).$$

Finally, the top edge boundary condition requires that

$$f_2(x) = u(x, b) = \sum_{n=1}^{\infty} B_n \sinh(\mu_n b) \sin(\mu_n x).$$

Conclusion

Appealing to the formulae for sine series coefficients, we can now summarize our findings.

Theorem

If $f_2(x)$ is piecewise smooth, the solution to the Dirichlet problem

$$\begin{aligned} \nabla^2 u &= 0, && 0 < x < a,\ 0 < y < b,\\ u(x,0) = 0,\ u(x,b) &= f_2(x), && 0 < x < a,\\ u(0,y) = u(a,y) &= 0, && 0 < y < b, \end{aligned}$$

is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin(\mu_n x)\sinh(\mu_n y),$$

where $\mu_n = \dfrac{n\pi}{a}$ *and* $B_n = \dfrac{2}{a\sinh(\mu_n b)} \displaystyle\int_0^a f_2(x)\sin(\mu_n x)\,dx.$

Remark: If we know the sine series expansion for $f_2(x)$ on $[0, a]$, then we can use the relationship

$$B_n = \frac{1}{\sinh(\mu_n b)} \left(n\text{th sine coefficient of } f_2\right).$$

Example

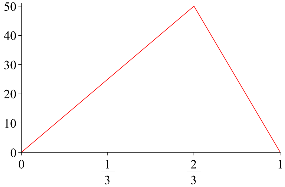
Solve the Dirichlet problem on the square $[0,1] \times [0,1]$, *subject to the boundary conditions*

$$\begin{aligned} u(x,0) &= 0,\ u(x,1) = f_2(x), && 0 < x < 1, \\ u(0,y) &= u(1,y) = 0, && 0 < y < 1, \end{aligned}$$

where

$$f_2(x) = \begin{cases} 75x & \text{if } 0 \le x \le \frac{2}{3}, \\ 150(1-x) & \text{if } \frac{2}{3} < x \le 1. \end{cases}$$

We have $a = b = 1$. The graph of $f_2(x)$ is:



According to exercise 2.4.17 (with $p = 1$, $a = 2/3$ and $h = 50$), the sine series for f_2 is:

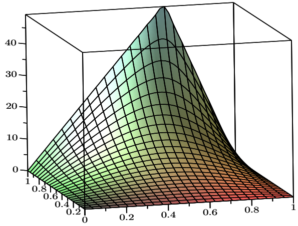
$$f_2(x) = \frac{450}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2} \sin(n\pi x).$$

Thus,

$$B_n = \frac{1}{\sinh(n\pi)}\left(\frac{450}{\pi^2}\frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2}\right) = \frac{450}{\pi^2}\frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2\sinh(n\pi)},$$

and

$$u(x,y) = \frac{450}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2\sinh(n\pi)}\sin(n\pi x)\sinh(n\pi y).$$



Solution of the Dirichlet problem on a rectangle
Cases A and B

We have already seen that the solution to (B) is given by

$$u_2(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right),$$

where

$$B_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f_2(x) \sin\left(\frac{n\pi x}{a}\right)\, dx.$$

Separation of variables to shows that the solution to (A) is

$$u_1(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi (b-y)}{a}\right),$$

where

$$A_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f_1(x) \sin\left(\frac{n\pi x}{a}\right)\, dx.$$

Likewise, the solution to (C) is

$$u_3(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi(a-x)}{b}\right) \sin\left(\frac{n\pi y}{b}\right),$$

with

$$C_n = \frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b g_1(y) \sin\left(\frac{n\pi y}{b}\right)\, dy.$$

And the solution to (D) is

$$u_4(x,y) = \sum_{n=1}^{\infty} D_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right),$$

where

$$D_n = \frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b g_2(y) \sin\left(\frac{n\pi y}{b}\right)\, dy.$$

Remarks:

- In each case, the coefficients of the solution are just multiples of the Fourier sine coefficients of the nonzero boundary condition, e.g.

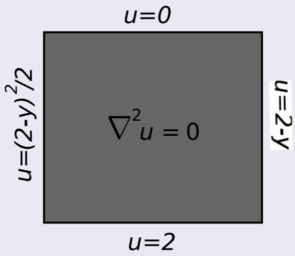
$$D_n = \frac{1}{\sinh\left(\frac{n\pi a}{b}\right)} \left(n\text{th sine coefficient of } g_2 \text{ on } [0,b]\right).$$

- The coefficients for each boundary condition are independent of the others.

- If any of the boundary conditions is zero, we may omit that term from the solution, e.g. if $g_1 \equiv 0$, then we don't need to include u_3.

Example

Solve the Dirichlet problem on $[0,1] \times [0,2]$ *with the following boundary conditions.*



We have $a = 1$, $b = 2$ and

$$f_1(x) = 2, \quad f_2(x) = 0, \quad g_1(y) = \frac{(2-y)^2}{2}, \quad g_2(y) = 2 - y.$$

It follows that $B_n = 0$ for all n, and the remaining coefficients we need are

$$A_n = \frac{2}{1 \cdot \sinh\left(\frac{n\pi 2}{1}\right)} \int_0^1 2\sin\left(\frac{n\pi x}{1}\right)\, dx = \frac{4(1+(-1)^{n+1})}{n\pi \sinh(2n\pi)},$$

$$C_n = \frac{2}{2\sinh\left(\frac{n\pi 1}{2}\right)} \int_0^2 \frac{(2-y)^2}{2} \sin\left(\frac{n\pi y}{2}\right)\, dy = \frac{4(\pi^2 n^2 - 2 + 2(-1)^n)}{n^3\pi^3 \sinh\left(\frac{n\pi}{2}\right)},$$

$$D_n = \frac{2}{2\sinh\left(\frac{n\pi 1}{2}\right)} \int_0^2 (2-y) \sin\left(\frac{n\pi y}{2}\right)\, dy = \frac{4}{n\pi \sinh\left(\frac{n\pi}{2}\right)}.$$

The complete solution is thus

$$
\begin{aligned}
u(x,y) = {} & \sum_{n=1}^{\infty} \frac{4(1+(-1)^{n+1})}{n\pi \sinh(2n\pi)} \sin(n\pi x) \sinh(n\pi(2-y)) \\
& + \sum_{n=1}^{\infty} \frac{4(n^2\pi^2 - 2 + 2(-1)^n)}{n^3\pi^3 \sinh\left(\frac{n\pi}{2}\right)} \sinh\left(\frac{n\pi(1-x)}{2}\right) \sin\left(\frac{n\pi y}{2}\right) \\
& + \sum_{n=1}^{\infty} \frac{4}{n\pi \sinh\left(\frac{n\pi}{2}\right)} \sinh\left(\frac{n\pi x}{2}\right) \sin\left(\frac{n\pi y}{2}\right).
\end{aligned}
$$