The Laplacian in Polar Coordinates

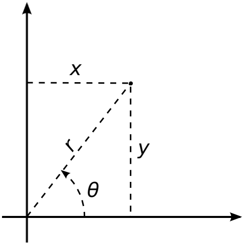
R. C. Daileda

Trinity University

Partial Differential Equations
March 20, 2014

Polar coordinates

To solve boundary value problems on circular regions, it is convenient to switch from rectangular (x, y) to polar (r, θ) spatial coordinates:



$$x = r\cos\theta,$$

$$y = r\sin\theta,$$

$$x^2 + y^2 = r^2.$$

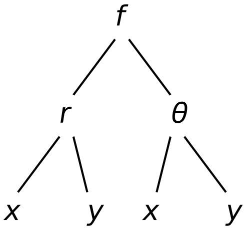
This requires us to express the rectangular Laplacian

$$\nabla^2 u = u_{xx} + u_{yy}$$

in terms of derivatives with respect to r and θ.

The chain rule

For any function $f(r,\theta)$, we have the familiar tree diagram and chain rule formulae:



$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}$$

or

$$f_x = f_r r_x + f_\theta \theta_x$$

$$f_y = f_r r_y + f_\theta \theta_y$$

First take $f = u$ to obtain

$$u_x = u_r r_x + u_\theta \theta_x \quad \Rightarrow \quad u_{xx} = u_r r_{xx} + (u_r)_x r_x + u_\theta \theta_{xx} + (u_\theta)_x \theta_x.$$

Applying the chain rule with $f = u_r$ and then with $f = u_\theta$ yields

$$\begin{aligned} u_{xx} &= u_r r_{xx} + \left(u_{rr} r_x + u_{r\theta}\theta_x\right) r_x + u_\theta \theta_{xx} + \left(u_{\theta r} r_x + u_{\theta\theta}\theta_x\right)\theta_x \\ &= u_r r_{xx} + u_{rr} r_x^2 + 2u_{r\theta} r_x \theta_x + u_\theta \theta_{xx} + u_{\theta\theta}\theta_x^2. \end{aligned}$$

An entirely similar computation using y instead of x also gives

$$u_{yy} = u_r r_{yy} + u_{rr} r_y^2 + 2u_{r\theta} r_y \theta_y + u_\theta \theta_{yy} + u_{\theta\theta}\theta_y^2.$$

If we add these expressions and collect like terms we get

$$\begin{aligned} \nabla^2 u = u_r \left(r_{xx} + r_{yy}\right) + u_{rr}\left(r_x^2 + r_y^2\right) + 2u_{r\theta}\left(r_x\theta_x + r_y\theta_y\right) \\ + u_\theta\left(\theta_{xx} + \theta_{yy}\right) + u_{\theta\theta}\left(\theta_x^2 + \theta_y^2\right). \end{aligned}$$

Differentiate $x^2 + y^2 = r^2$ with respect to x and then y:

$$2x = 2rr_x \;\Rightarrow\; r_x = \frac{x}{r} \;\Rightarrow\; r_{xx} = \frac{r - xr_x}{r^2} = \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3},$$

$$2y = 2rr_y \;\Rightarrow\; r_y = \frac{y}{r} \;\Rightarrow\; r_{yy} = \frac{r - yr_y}{r^2} = \frac{r^2 - y^2}{r^3} = \frac{x^2}{r^3}.$$

Now differentiate $\tan\theta = \frac{y}{x}$ with respect to x and then y:

$$\sec^2\theta\,\theta_x = -\frac{y}{x^2} \;\Rightarrow\; \theta_x = -\frac{y\cos^2\theta}{x^2} = -\frac{y}{r^2} \;\Rightarrow\; \theta_{xx} = \frac{2y}{r^3}r_x = \frac{2xy}{r^4},$$

$$\sec^2\theta\,\theta_y = \frac{1}{x} \;\Rightarrow\; \theta_y = \frac{\cos^2\theta}{x} = \frac{x}{r^2} \;\Rightarrow\; \theta_{yy} = \frac{-2x}{r^3}r_y = -\frac{2xy}{r^4}.$$

Together these yield

$$r_{xx} + r_{yy} = \frac{y^2 + x^2}{r^3} = \frac{1}{r}, \quad r_x^2 + r_y^2 = \frac{x^2 + y^2}{r^2} = 1.$$

$$\theta_{xx} + \theta_{yy} = \frac{2xy}{r^4} + \frac{-2xy}{r^4} = 0, \quad \theta_x^2 + \theta_y^2 = \frac{y^2 + x^2}{r^4} = \frac{1}{r^2},$$

$$r_x\theta_x + r_y\theta_y = \frac{-xy}{r^3} + \frac{yx}{r^3} = 0,$$

and we finally obtain

$$\begin{aligned}\nabla^2 u = & u_r\left(r_{xx} + r_{yy}\right) + u_{rr}\left(r_x^2 + r_y^2\right) + 2u_{r\theta}\left(r_x\theta_x + r_y\theta_y\right) \\ & + u_\theta\left(\theta_{xx} + \theta_{yy}\right) + u_{\theta\theta}\left(\theta_x^2 + \theta_y^2\right) \\ = & \frac{1}{r}u_r + u_{rr} + \frac{1}{r^2}u_{\theta\theta} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.\end{aligned}$$

Example

Use polar coordinates to show that the function $u(x,y) = \dfrac{y}{x^2+y^2}$ *is harmonic.*

We need to show that $\nabla^2 u = 0$. In polar coordinates we have

$$u(r,\theta) = \frac{r\sin\theta}{r^2} = \frac{\sin\theta}{r}$$

so that

$$u_r = -\frac{\sin\theta}{r^2}, \qquad u_{rr} = \frac{2\sin\theta}{r^3}, \qquad u_{\theta\theta} = \frac{-\sin\theta}{r},$$

and thus

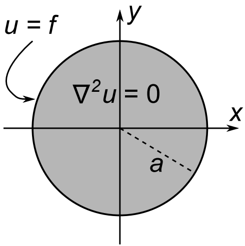
$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \frac{2\sin\theta}{r^3} - \frac{\sin\theta}{r^3} - \frac{\sin\theta}{r^3} = 0.$$

The Dirichlet problem on a disk

Goal: Solve the Dirichlet problem on a disk of radius a, centered at the origin. In polar coordinates this has the form

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad 0 \leq r < a,$$

$$u(a, \theta) = f(\theta), \quad 0 \leq \theta \leq 2\pi.$$



Remarks:

- We will require that f is 2π-periodic.
- Likewise, we require that $u(r, \theta)$ is 2π-periodic in θ.

Separation of variables

If we assume that $u(r,\theta) = R(r)\Theta(\theta)$ and plug into $\nabla^2 u = 0$, we get

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \;\Rightarrow\; r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$$

$$\Rightarrow\; r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

This yields the pair of separated ODEs

$$r^2R'' + rR' - \lambda R = 0 \quad \text{and} \quad \Theta'' + \lambda\Theta = 0.$$

We also have the "boundary conditions"

$$\Theta \text{ is } 2\pi\text{-periodic} \quad \text{and} \quad \lim_{r\to 0^+} R(r) \text{ is finite.}$$

Solving for Θ

The solutions of $\Theta'' + \lambda\Theta = 0$ are periodic only if

$$\lambda = \mu^2 \geq 0 \quad \Rightarrow \quad \Theta = a\cos(\mu\theta) + b\sin(\mu\theta).$$

In order for the period to be 2π we also need

$$1 = \cos(0\mu) = \cos(2\pi\mu) \quad \Rightarrow \quad 2\pi\mu = 2\pi n \quad \Rightarrow \quad \mu = n \in \mathbb{N}_0.$$

Hence $\lambda = n^2$ and

$$\Theta = \Theta_n = a_n\cos(n\theta) + b_n\sin(n\theta), \quad n \in \mathbb{N}_0.$$

It follows that R satisfies

$$r^2R'' + rR' - n^2R = 0,$$

which is called an *Euler equation*.

An *Euler equation* is a second order ODE of the form

$$x^2y'' + \alpha xy' + \beta y = 0.$$

Its solutions are determined by the roots of its *indicial equation*

$$\rho^2 + (\alpha - 1)\rho + \beta = 0.$$

Case 1: If the roots are $\rho_1 \neq \rho_2$, then the general solution is

$$y = c_1 x^{\rho_1} + c_2 x^{\rho_2}.$$

Case 2: If there is only one root ρ_1, then the general solution is

$$y = c_1 x^{\rho_1} + c_2 x^{\rho_1} \ln x.$$

Solving for R

The indicial equation of $r^2R'' + rR' - n^2R = 0$ is

$$\rho^2 + (1-1)\rho - n^2 = \rho^2 - n^2 = 0 \ \Rightarrow \ \rho = \pm n.$$

This means that

$$\begin{aligned} R &= c_1 r^n + c_2 r^{-n} \quad (n \neq 0), \\ R &= c_1 + c_2 \ln r \qquad (n = 0). \end{aligned}$$

These will be finite at $r = 0$ only if $c_2 = 0$. Setting $c_1 = a^{-n}$ gives

$$R = R_n = \left(\frac{r}{a}\right)^n, \quad n \in \mathbb{N}_0.$$

Separated solutions and superposition

We therefore obtain the separated solutions

$$u_n(r,\theta) = R_n(r)\Theta_n(\theta) = \left(\frac{r}{a}\right)^n (a_n \cos(n\theta) + b_n \sin(n\theta)), \quad n \in \mathbb{N}_0.$$

Noting that

$$u_0(r,\theta) = \left(\frac{r}{a}\right)^0 (a_0 \cos 0 + b_0 \sin 0) = a_0,$$

superposition gives the general solution

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

Boundary values and conclusion

Imposing our Dirichlet boundary conditions gives

$$f(\theta) = u(a,\theta) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right),$$

which is just the ordinary 2π-periodic Fourier series for f!

Theorem

The solution of the Dirichlet problem on the disk of radius a centered at the origin, with boundary condition $u(a,\theta) = f(\theta)$ is $u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)$, where

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta,$$

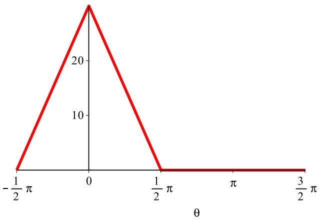
$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta)\, d\theta, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta)\, d\theta.$$

Example

Find the solution to the Dirichlet problem on a disk of radius 3 with boundary values given by

$$f(\theta) = \begin{cases} \frac{30}{\pi}(\pi + 2\theta) & \text{if } \frac{-\pi}{2} \leq \theta < 0, \\ \frac{30}{\pi}(\pi - 2\theta) & \text{if } 0 \leq \theta < \frac{\pi}{2}, \\ 0 & \text{if } \frac{\pi}{2} \leq \theta < \frac{3\pi}{2}. \end{cases}$$

We have $a = 3$. The graph of f is

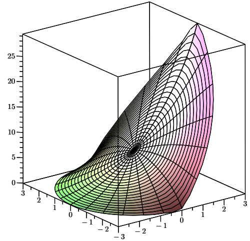


According to exercise 2.3.8 (with $p = \pi$, $c = 30$ and $d = \pi/2$):

$$f(\theta) = \frac{15}{2} + \frac{120}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - \cos(n\pi/2)}{n^2} \cos(n\theta).$$

Hence, the solution to the Dirichlet problem is

$$u(r,\theta) = \frac{15}{2} + \frac{120}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{r}{3}\right)^n \frac{1 - \cos(n\pi/2)}{n^2} \cos(n\theta).$$



Example

Solve the Dirichlet problem on a disk of radius 2 with boundary values given by $f(\theta) = \cos^2\theta$. Express your answer in cartesian coordinates.

We have $a = 2$ and

$$f(\theta) = \cos^2\theta = \frac{1+\cos(2\theta)}{2} = \frac{1}{2} + \frac{1}{2}\cos(2\theta),$$

which is a finite 2π-periodic Fourier series (i.e. $a_0 = 1/2$, $a_2 = 1/2$, and all other coefficients are zero).
Hence

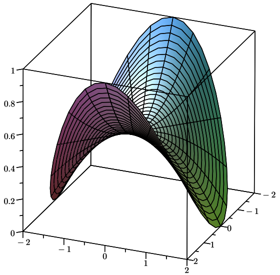
$$u(r,\theta) = \frac{1}{2} + \left(\frac{r}{2}\right)^2 \cdot \frac{1}{2}\cos(2\theta) = \frac{1}{2} + \frac{r^2\cos(2\theta)}{8}.$$

Since $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, we find that

$$r^2\cos(2\theta) = r^2\cos^2\theta - r^2\sin^2\theta = x^2 - y^2$$

and hence

$$u = \frac{1}{2} + \frac{r^2\cos(2\theta)}{8} = \frac{1}{2} + \frac{x^2 - y^2}{8}.$$



Example

Solve the Dirichlet problem on a disk of radius 1 if the boundary value is 50 in the first quadrant, and zero elsewhere.

We are given $a = 1$, $f(\theta) = 50$ for $0 \leq \theta \leq \pi/2$ and $f(\theta) = 0$ otherwise. The Fourier coefficients of f are

$$a_0 = \frac{1}{2\pi}\int_0^{\pi/2} 50\, d\theta = \frac{25}{2},$$

$$a_n = \frac{1}{\pi}\int_0^{\pi/2} 50\cos(n\theta)\, d\theta = \frac{50\sin(n\pi/2)}{n\pi},$$

$$b_n = \frac{1}{\pi}\int_0^{\pi/2} 50\sin(n\theta)\, d\theta = \frac{50(1-\cos(n\pi/2))}{n\pi},$$

so that

$$u(r,\theta) = \frac{25}{2} + \frac{50}{\pi}\sum_{n=1}^{\infty} r^n \left(\frac{\sin(n\pi/2)}{n}\cos(n\theta) + \frac{(1-\cos(n\pi/2))}{n}\sin(n\theta)\right).$$

Remarks:

- One can frequently use identities like (valid for $|r| < 1$)

$$\sum_{n=1}^{\infty} \frac{r^n \cos(n\theta)}{n} = -\frac{1}{2} \ln\left(1 - 2r\cos\theta + r^2\right),$$
$$\sum_{n=1}^{\infty} \frac{r^n \sin(n\theta)}{n} = \arctan\left(\frac{r\sin\theta}{1 - r\cos\theta}\right),$$

  to convert series solutions in polar coordinates to cartesian expressions.

- Using the second identity, one can show that the solution in the preceding example is

$$u(x,y) = \frac{25}{2} + \frac{50}{\pi}\left(\arctan\left(\frac{y}{1-x}\right) + \arctan\left(\frac{x}{1-y}\right)\right).$$