An Introduction to Bessel Functions

R. C. Daileda

Trinity University

Partial Differential Equations
March 25, 2014

Bessel's equation

Given $p \geq 0$, the ordinary differential equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \quad x > 0 \tag{1}$$

is known as *Bessel's equation of order p*.

- Solutions to (1) are known as *Bessel functions.*
- Since (1) is a second order homogeneous linear equation, the general solution is a linear combination of any two linearly independent (i.e. *fundamental*) solutions.
- We will describe and give the basic properties of the most commonly used pair of fundamental solutions.

The method of Frobenius

We begin by assuming the solution has the form

$$y = \sum_{m=0}^{\infty} a_m x^{r+m} \qquad (a_0 \neq 0)$$

and try to determine r and a_m.

Substituting into Bessel's equation and collecting terms with common powers of x gives

$$a_0(r^2 - p^2)x^r + a_1\left((r+1)^2 - p^2\right)x^{r+1} + \sum_{m=2}^{\infty}\left(a_m\left((r+m)^2 - p^2\right) + a_{m-2}\right)x^{r+m} = 0.$$

Setting the coefficients equal to zero gives the equations

$$a_0(r^2 - p^2) = 0 \quad \Rightarrow \quad r = \pm p, \quad a_1\left((r+1)^2 - p^2\right) = 0 \quad \Rightarrow \quad a_1 = 0,$$

$$a_m\left((r+m)^2 - p^2\right) + a_{m-2} = 0$$

$$\Rightarrow \quad a_m = \frac{-a_{m-2}}{(r+m)^2 - p^2} = \frac{-a_{m-2}}{m(m+2p)} \quad (m \geq 2).$$

This means that $a_1 = a_3 = a_5 = \cdots = a_{2k+1} = 0$ and

$$a_2 = \frac{-a_0}{2(2+2p)} = \frac{-a_0}{2^2(1+p)},$$

$$a_4 = \frac{-a_2}{4(4+2p)} = \frac{-a_2}{2^2 2(2+p)} = \frac{a_0}{2^4 2(1+p)(2+p)},$$

$$a_6 = \frac{-a_4}{6(6+2p)} = \frac{-a_4}{2^2 3(3+p)} = \frac{-a_0}{2^6 3!(1+p)(2+p)(3+p)},$$

$$a_8 = \frac{-a_6}{8(8+2p)} = \frac{-a_6}{2^2 4(4+p)} = \frac{a_0}{2^8 4!(1+p)(2+p)(3+p)(4+p)}.$$

In general, we see that

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k!(1+p)(2+p)\cdots(k+p)}.$$

Setting $r = p$ and $m = 2k$ in the original series gives

$$\begin{aligned} y &= \sum_{k=0}^{\infty} \frac{(-1)^k a_0}{2^{2k} k!(1+p)(2+p)\cdots(k+p)} x^{2k+p} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^p a_0}{k!(1+p)(2+p)\cdots(k+p)} \left(\frac{x}{2}\right)^{2k+p}. \end{aligned}$$

The standard way to choose a_0 involves the so-called *Gamma function.*

Interlude
The Gamma function

The *Gamma function* is defined to be

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \qquad (x > 0).$$

One can use integration by parts to show that

$$\Gamma(x+1) = x\,\Gamma(x).$$

Applying this repeatedly, we find that for $k \in \mathbb{N}$

$$\begin{aligned}
\Gamma(x+k) &= (x+k-1)\Gamma(x+k-1) \\
&= (x+k-1)(x+k-2)\Gamma(x+k-2) \\
&= (x+k-1)(x+k-2)(x+k-3)\Gamma(x+k-3) \\
&\ \vdots \\
&= (x+k-1)(x+k-2)(x+k-3)\cdots x\,\Gamma(x).
\end{aligned}$$

This has two nice consequences.

- According to the definition, one has $\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$. Setting $x = 1$ above:

$$\Gamma(k+1) = k(k-1)(k-2)\cdots 1 \cdot \Gamma(1) = k!$$

This is why $\Gamma(x)$ is called the *generalized factorial*.

- Setting $x = p+1$ above:

$$\Gamma(p+1+k) = (p+k)(p+k-1)\cdots(p+1)\Gamma(p+1)$$

or

$$\frac{1}{(1+p)(2+p)\cdots(k+p)} = \frac{\Gamma(p+1)}{\Gamma(k+p+1)}.$$

Bessel functions of the first and second kind

Returning to Bessel's equation,

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \qquad x > 0$$

choosing $a_0 = \dfrac{1}{2^p\Gamma(p+1)}$ in the Frobenius solution, we now see that

$$y = J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p},$$

is one solution.

$J_p(x)$ is called the *Bessel function of the first kind of order p*.

Remarks

- A second linearly independent solution can be found via *reduction of order*. When (appropriately normalized), it is denoted by

$$Y_p(x),$$

  and is called the *Bessel function of the second kind of order p*.

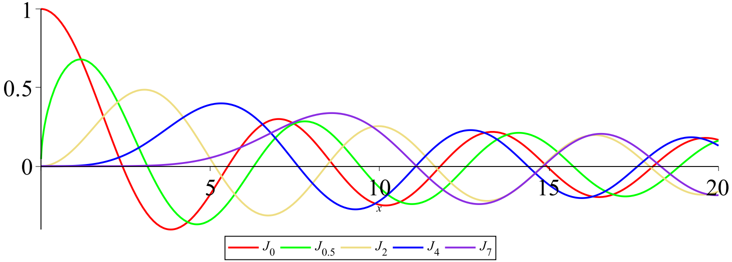
- The general solution to Bessel's equation is

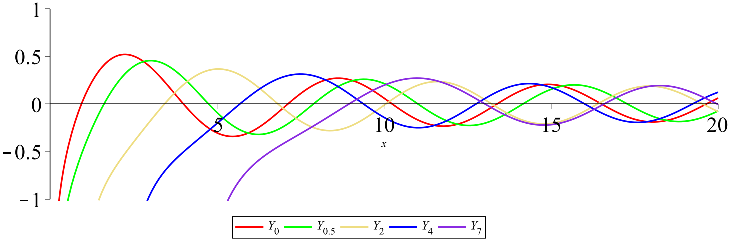
$$y = c_1 J_p(x) + c_2 Y_p(x).$$

- In Maple, the functions $J_p(x)$ and $Y_p(x)$ are called by the commands

`BesselJ(p,x)` and `BesselY(p,x)`.

Graphs of Bessel functions





Properties of Bessel functions

- $J_0(0) = 1$, $J_p(0) = 0$ for $p > 0$ and $\lim_{x \to 0^+} Y_p(x) = -\infty$.
- The values of J_p always lie between 1 and -1.
- J_p has infinitely many positive zeros, which we denote by

$$0 < \alpha_{p1} < \alpha_{p2} < \alpha_{p3} < \cdots$$

- J_p is oscillatory and tends to zero as $x \to \infty$. More precisely,

$$J_p(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{p\pi}{2} - \frac{\pi}{4}\right).$$

- $\lim_{n \to \infty} |\alpha_{pn} - \alpha_{p,n+1}| = \pi$.

- For $0 < p < 1$, the graph of J_p has a vertical tangent line at $x = 0$.
- For $1 < p$, the graph of J_p has a horizontal tangent line at $x = 0$, and the graph is initially "flat."
- For some values of p, the Bessel functions of the first kind can be expressed in terms of familiar functions, e.g.

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x,$$
$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right).$$

Differentiation identities

Using the series definition of $J_p(x)$, one can show that:

$$\begin{aligned} \frac{d}{dx}\left(x^p J_p(x)\right) &= x^p J_{p-1}(x), \\ \frac{d}{dx}\left(x^{-p} J_p(x)\right) &= -x^{-p} J_{p+1}(x). \end{aligned} \tag{2}$$

The product rule and cancellation lead to

$$\begin{aligned} xJ_p'(x) + pJ_p(x) &= xJ_{p-1}(x), \\ xJ_p'(x) - pJ_p(x) &= -xJ_{p+1}(x). \end{aligned}$$

Addition and subtraction of these identities then yield

$$\begin{aligned} J_{p-1}(x) - J_{p+1}(x) &= 2J_p'(x), \\ J_{p-1}(x) + J_{p+1}(x) &= \frac{2p}{x} J_p(x). \end{aligned}$$

Integration identities

Integration of the differentiation identities (2) gives

$$\int x^{p+1} J_p(x)\,dx = x^{p+1} J_{p+1}(x) + C$$
$$\int x^{-p+1} J_p(x)\,dx = -x^{-p+1} J_{p-1}(x) + C.$$

- Exercises 4.2.12 and 4.3.9 give similar identities.
- Identities such as these can be used to evaluate certain integrals of the form

$$\int_0^a f(r) J_m(\lambda_{mn} r) r\,dr,$$

which will occur frequently in later work.

Example

Evaluate

$$\int x^{p+5} J_p(x)\,dx.$$

We integrate by parts, first taking

$$\begin{aligned} u &= x^4 & dv &= x^{p+1} J_p(x)\,dx \\ du &= 4x^3\,dx & v &= x^{p+1} J_{p+1}(x), \end{aligned}$$

which gives

$$\int x^{p+5} J_p(x)\,dx = x^{p+5} J_{p+1}(x) - 4 \int x^{p+4} J_{p+1}(x)\,dx.$$

Now integrate by parts again with

$$u = x^2 \qquad dv = x^{p+2} J_{p+1}(x)\, dx$$
$$du = 2x\, dx \qquad v = x^{p+2} J_{p+2}(x),$$

to get

$$\begin{aligned}\int x^{p+5} J_p(x)\, dx &= x^{p+5} J_{p+1}(x) - 4 \int x^{p+4} J_{p+1}(x)\, dx \\ &= x^{p+5} J_{p+1}(x) - 4\left(x^{p+4} J_{p+2}(x) - 2\int x^{p+3} J_{p+2}(x)\, dx\right) \\ &= x^{p+5} J_{p+1}(x) - 4x^{p+4} J_{p+2}(x) + 8x^{p+3} J_{p+3}(x) + C.\end{aligned}$$

The parametric form of Bessel's equation

For $p \geq 0$, consider the *parametric Bessel equation*

$$x^2 y'' + xy' + (\lambda^2 x^2 - p^2)y = 0 \quad (\lambda > 0). \tag{3}$$

If we let $\xi = \lambda x$, then the chain rule implies

$$\begin{aligned} y' &= \frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = \lambda \dot{y}, \\ y'' &= \frac{dy'}{dx} = \lambda \frac{d\dot{y}}{dx} = \lambda \frac{d\dot{y}}{d\xi}\frac{d\xi}{dx} = \lambda^2 \ddot{y}. \end{aligned}$$

Hence (3) becomes

$$\xi^2 \ddot{y} + \xi \dot{y} + (\xi^2 - p^2)y = 0,$$

which is Bessel's equation in the variable ξ.

It follows that

$$y = c_1 J_p(\xi) + c_2 Y_p(\xi) = c_1 J_p(\lambda x) + c_2 Y_p(\lambda x)$$

gives the general solution to the parametric Bessel equation.

Because $\lim_{x \to 0^+} Y_p(x) = -\infty$, we find that

$$y(0) \text{ is finite} \quad \Rightarrow \quad c_2 = 0,$$

so that the only solutions that are *defined at* $x = 0$ are

$$y = c_1 J_p(\lambda x).$$

This will be important in later work.