

# The two dimensional wave equation

R. C. Daileda



Trinity University

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# Vibrating membranes

**Goal:** Model the motion of an ideal elastic membrane.

**Set up:** Assume the membrane at rest is a region of the  $xy$ -plane and let

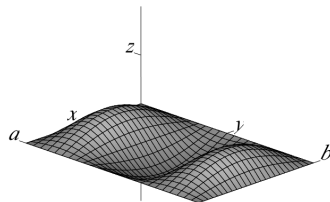
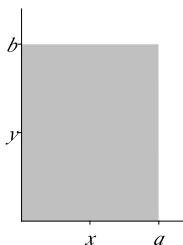
$u(x, y, t)$  = vertical deflection of membrane from equilibrium at position  $(x, y)$  and time  $t$ .

For a fixed  $t$ , the surface  $z = u(x, y, t)$  gives the shape of the membrane at time  $t$ .

Under ideal assumptions (e.g. uniform membrane density, uniform tension, no resistance to motion, small deflection, etc.) one can show that  $u$  satisfies the *two dimensional wave equation*

$$u_{tt} = c^2 \nabla^2 u = c^2 (u_{xx} + u_{yy}).$$

We assume the membrane lies over the rectangular region  $R = [0, a] \times [0, b]$  and has fixed edges.



These facts are expressed by the *boundary conditions*

$$\begin{aligned}u(0, y, t) = u(a, y, t) &= 0, & 0 \leq y \leq b, t > 0, \\u(x, 0, t) = u(x, b, t) &= 0, & 0 \leq x \leq a, t > 0.\end{aligned}$$

We must also specify how the membrane is initially deformed and set into motion. This is done via the *initial conditions*

$$\begin{aligned}u(x, y, 0) &= f(x, y), & (x, y) &\in R, \\u_t(x, y, 0) &= g(x, y), & (x, y) &\in R.\end{aligned}$$

**New goal:** solve the 2-D wave equation subject to the boundary and initial conditions just given.

We will:

- Use separation of variables to find simple solutions satisfying the homogeneous boundary conditions; and
- Use the principle of superposition to build up a series solution that satisfies the initial conditions as well.

# Separation of variables

We seek nontrivial solutions of the form

$$u(x, y, t) = X(x)Y(y)T(t).$$

Plugging this into  $u_{tt} = c^2(u_{xx} + u_{yy})$  we get

$$XYT'' = c^2(X''YT + XY''T) \Rightarrow \frac{T''}{c^2T} = \frac{X''}{X} + \frac{Y''}{Y}.$$

Because the two sides are functions of different independent variables, they must be constant:

$$\frac{T''}{c^2T} = A = \frac{X''}{X} + \frac{Y''}{Y} \Rightarrow \begin{cases} T'' - c^2AT = 0, \\ \frac{X''}{X} = -\frac{Y''}{Y} + A. \end{cases}$$

Since the two sides again involve unrelated variables, both are constant:

$$\frac{X''}{X} = B = -\frac{Y''}{Y} + A.$$

Setting  $C = A - B$ , these equations can be rewritten as

$$X'' - BX = 0, \quad Y'' - CY = 0.$$

The first boundary condition is

$$0 = u(0, y, t) = X(0)Y(y)T(t).$$

Canceling  $Y$  and  $T$  yields  $X(0) = 0$ . Likewise, we obtain

$$X(a) = 0, \quad Y(0) = Y(b) = 0.$$

There are no boundary conditions on  $T$ .

We have already solved the two boundary value problems for  $X$  and  $Y$ . The nontrivial solutions are

$$X = X_m(x) = \sin(\mu_m x), \quad \mu_m = \frac{m\pi}{a}, \quad m \in \mathbb{N},$$

$$Y = Y_n(y) = \sin(\nu_n y), \quad \nu_n = \frac{n\pi}{b}, \quad n \in \mathbb{N},$$

with separation constants  $B = -\mu_m^2$  and  $C = -\nu_n^2$ .

Since  $T'' - c^2 AT = 0$ , and  $A = B + C = -(\mu_m^2 + \nu_n^2) < 0$ ,

$$T = T_{mn}(t) = B_{mn} \cos(\lambda_{mn} t) + B_{mn}^* \sin(\lambda_{mn} t),$$

where

$$\lambda_{mn} = c \sqrt{\mu_m^2 + \nu_n^2} = c\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

These are the **characteristic frequencies** of the membrane.

# Normal modes and Superposition

Assembling our results, we find that for any pair  $m, n \in \mathbb{N}$  we have the *normal mode*

$$\begin{aligned}u_{mn}(x, y, t) &= X_m(x)Y_n(y)T_{mn}(t) \\ &= \sin(\mu_m x) \sin(\nu_n y) (B_{mn} \cos(\lambda_{mn} t) + B_{mn}^* \sin(\lambda_{mn} t))\end{aligned}$$

where

$$\mu_m = \frac{m\pi}{a}, \quad \nu_n = \frac{n\pi}{b}, \quad \lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2}.$$

Superposition gives the general solution

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin(\mu_m x) \sin(\nu_n y) (B_{mn} \cos(\lambda_{mn} t) + B_{mn}^* \sin(\lambda_{mn} t)).$$



## Remarks:

- Note that the normal modes:
  - oscillate spatially with frequency  $\mu_m$  in the  $x$ -direction,
  - oscillate spatially with frequency  $\nu_n$  in the  $y$ -direction,
  - oscillate temporally with frequency  $\lambda_{mn}$ .
- While  $\mu_m$  and  $\nu_n$  are simply multiples of  $\pi/a$  and  $\pi/b$ , respectively,  
 *$\lambda_{mn}$  is not a multiple of any basic frequency.*
- We must use a double series in the general solution because  $m$  and  $n$  are independent.

## Initial conditions

Finally, we impose the initial conditions to determine the coefficients  $B_{mn}$  and  $B_{mn}^*$ . These yield

$$f(x, y) = u(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right),$$

$$g(x, y) = u_t(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lambda_{mn} B_{mn}^* \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right).$$

These are examples of *double Fourier series*.

As before, we are faced with two **Questions**:

- Which functions are given by double Fourier series?
- How can we compute the coefficients in a double Fourier series?

# Representability

The following result partially answers the first question.

## Theorem

If  $f(x, y)$  is a  $C^2$  function on the rectangle  $[0, a] \times [0, b]$ , then

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right),$$

for appropriate  $B_{mn}$ .

- To say that  $f(x, y)$  is a  $C^2$  function means that  $f$  as well as its first and second order partial derivatives are all continuous.
- While not as general as the Fourier representation theorem, this result is sufficient for our applications.

## Orthogonality (again!)

The following result helps us answer the second question.

### Theorem

*The functions*

$$Z_{mn}(x, y) = \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right), \quad m, n \in \mathbb{N}$$

*are pairwise orthogonal relative to the inner product*

$$\langle f, g \rangle = \int_0^a \int_0^b f(x, y)g(x, y) dy dx.$$

This is easily verified using the orthogonality of the functions  $\sin(n\pi x/p)$  on the interval  $[0, p]$ .

Using the usual argument, it follows that if

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{mn} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right),$$

then

$$\begin{aligned} B_{mn} &= \frac{\langle f, Z_{mn} \rangle}{\langle Z_{mn}, Z_{mn} \rangle} = \frac{\int_0^a \int_0^b f(x, y) Z_{mn}(x, y) dy dx}{\int_0^a \int_0^b Z_{mn}(x, y)^2 dy dx} \\ &= \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) dy dx. \end{aligned}$$

So, we can finally write down the complete solution to our original problem.

# Conclusion

## Theorem

Suppose that  $f(x, y)$  and  $g(x, y)$  are  $C^2$  functions on the rectangle  $[0, a] \times [0, b]$ . The solution to the vibrating membrane problem is given by  $u(x, y, t) =$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin(\mu_m x) \sin(\nu_n y) (B_{mn} \cos(\lambda_{mn} t) + B_{mn}^* \sin(\lambda_{mn} t))$$

where  $\mu_m = \frac{m\pi}{a}$ ,  $\nu_n = \frac{n\pi}{b}$ ,  $\lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2}$ , and

$$B_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin(\mu_m x) \sin(\nu_n y) dy dx,$$

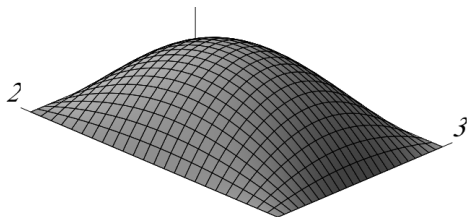
$$B_{mn}^* = \frac{4}{ab\lambda_{mn}} \int_0^a \int_0^b g(x, y) \sin(\mu_m x) \sin(\nu_n y) dy dx.$$

### Example

A  $2 \times 3$  rectangular membrane has  $c = 6$ . If we deform it to have shape given by

$$f(x, y) = xy(2 - x)(3 - y),$$

keep its edges fixed, and release it at  $t = 0$ , find an expression that gives the shape of the membrane for  $t > 0$ .



We must compute the coefficients  $B_{mn}$  and  $B_{mn}^*$ . Since  $g(x, y) = 0$  we immediately have

$$B_{mn}^* = 0.$$

We also have

$$\begin{aligned} B_{mn} &= \frac{4}{2 \cdot 3} \int_0^2 \int_0^3 xy(2-x)(3-y) \sin\left(\frac{m\pi}{2}x\right) \sin\left(\frac{n\pi}{3}y\right) dy dx \\ &= \frac{2}{3} \int_0^2 x(2-x) \sin\left(\frac{m\pi}{2}x\right) dx \int_0^3 y(3-y) \sin\left(\frac{n\pi}{3}y\right) dy \\ &= \frac{2}{3} \left( \frac{16(1 + (-1)^{m+1})}{\pi^3 m^3} \right) \left( \frac{54(1 + (-1)^{n+1})}{\pi^3 n^3} \right) \\ &= \frac{576}{\pi^6} \frac{(1 + (-1)^{m+1})(1 + (-1)^{n+1})}{m^3 n^3}. \end{aligned}$$



The coefficients  $\lambda_{mn}$  are given by

$$\lambda_{mn} = 6\pi \sqrt{\frac{m^2}{4} + \frac{n^2}{9}} = \pi \sqrt{9m^2 + 4n^2}.$$

Assembling all of these pieces yields

$$u(x, y, t) = \frac{576}{\pi^6} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \frac{(1 + (-1)^{m+1})(1 + (-1)^{n+1})}{m^3 n^3} \sin\left(\frac{m\pi}{2}x\right) \right. \\ \left. \times \sin\left(\frac{n\pi}{3}y\right) \cos\left(\pi \sqrt{9m^2 + 4n^2} t\right) \right).$$

### Example

Suppose in the previous example we also impose an initial velocity given by  $g(x, y) = \sin 2\pi x$ . Find an expression that gives the shape of the membrane for  $t > 0$ .

Since we have the same initial shape,  $B_{mn}$  don't change. We only need to find  $B_{mn}^*$  and add the appropriate terms to the previous solution.

Using  $\lambda_{mn}$  computed above, we have

$$\begin{aligned} B_{mn}^* &= \frac{2}{3\pi\sqrt{9m^2 + 4n^2}} \int_0^2 \int_0^3 \sin(2\pi x) \sin\left(\frac{m\pi}{2}x\right) \sin\left(\frac{n\pi}{3}y\right) dy dx \\ &= \frac{2}{3\pi\sqrt{9m^2 + 4n^2}} \int_0^2 \sin(2\pi x) \sin\left(\frac{m\pi}{2}x\right) dx \int_0^3 \sin\left(\frac{n\pi}{3}y\right) dy. \end{aligned}$$

The first integral is zero unless  $m = 4$ , i.e. and  $B_{mn}^* = 0$  for  $m \neq 4$ .

Evaluating the second integral, we have

$$B_{4n}^* = \frac{1}{3\pi\sqrt{36+n^2}} \frac{3(1+(-1)^{n+1})}{n\pi} = \frac{1+(-1)^{n+1}}{\pi^2 n\sqrt{36+n^2}}.$$

So the velocity dependent term of the solution is

$$\begin{aligned} u_2(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}^* \sin(\mu_m x) \sin(\nu_n y) \sin(\lambda_{mn} t) \\ &= \frac{\sin(2\pi x)}{\pi^2} \sum_{n=1}^{\infty} \frac{1+(-1)^{n+1}}{n\sqrt{36+n^2}} \sin\left(\frac{n\pi}{3} y\right) \sin\left(2\pi\sqrt{36+n^2} t\right). \end{aligned}$$

If we let  $u_1(x, y, t)$  denote the solution to the first example, the complete solution here is

$$u(x, y, t) = u_1(x, y, t) + u_2(x, y, t).$$