The two-dimensional heat equation

Ryan C. Daileda

Trinity University

Partial Differential Equations
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Physical motivation

Goal: Model heat flow in a two-dimensional object (thin plate).

Set up: Represent the plate by a region in the xy-plane and let

$u(x, y, t) =$ temperature of plate at position (x, y) and time t.

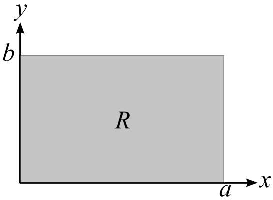
For a fixed t, the height of the surface $z = u(x, y, t)$ gives the temperature of the plate at time t and position (x, y).

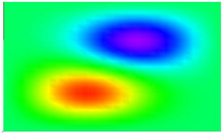
Under ideal assumptions (e.g. uniform density, uniform specific heat, perfect insulation along faces, no internal heat sources etc.) one can show that u satisfies the *two dimensional heat equation*

$$u_t = c^2 \nabla^2 u = c^2 (u_{xx} + u_{yy})$$

For now we assume:

- The plate is rectangular, represented by $R = [0, a] \times [0, b]$.





- The plate is imparted with some initial temperature:

$$u(x, y, 0) = f(x, y), \quad (x, y) \in R.$$

- The edges of the plate are held at zero degrees:

$$\begin{aligned} u(0, y, t) = u(a, y, t) = 0, &\qquad 0 \le y \le b,\ t > 0, \\ u(x, 0, t) = u(x, b, t) = 0, &\qquad 0 \le x \le a,\ t > 0. \end{aligned}$$

Separation of variables

Assuming that $u(x, y, t) = X(x)Y(y)T(t)$, and proceeding as we did with the 2-D wave equation, we find that

$$\begin{aligned} X'' - BX = 0, &\quad X(0) = X(a) = 0, \\ Y'' - CY = 0, &\quad Y(0) = Y(b) = 0, \\ T' - c^2(B + C)T = 0. & \end{aligned}$$

We have already solved the first two of these problems:

$$\begin{aligned} X = X_m(x) = \sin(\mu_m x), &\qquad \mu_m = \frac{m\pi}{a}, &\qquad B = -\mu_m^2 \\ Y = Y_n(y) = \sin(\nu_n y), &\qquad \nu_n = \frac{n\pi}{b}, &\qquad C = -\nu_n^2, \end{aligned}$$

for $m, n \in \mathbb{N}$. It then follows that

$$T = T_{mn}(t) = A_{mn} e^{-\lambda_{mn}^2 t}, \quad \lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

Superposition

Assembling these results, we find that for any *pair* $m, n \geq 1$ we have the normal mode

$$u_{mn}(x,y,t) = X_m(x)Y_n(y)T_{mn}(t) = A_{mn}\sin(\mu_m x)\,\sin(\nu_n y)\,e^{-\lambda_{mn}^2 t}.$$

The principle of superposition gives the general solution

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(\mu_m x)\,\sin(\nu_n y)\,e^{-\lambda_{mn}^2 t}.$$

The initial condition requires that

$$f(x,y) = u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin\left(\frac{m\pi}{a}x\right)\,\sin\left(\frac{n\pi}{b}y\right),$$

which is just the double Fourier series for $f(x,y)$.

Conclusion

Theorem

Suppose that $f(x, y)$ is a C^2 function on the rectangle $[0, a] \times [0, b]$. The solution to the heat equation with homogeneous Dirichlet boundary conditions and initial condition $f(x, y)$ is

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(\mu_m x) \sin(\nu_n y)\, e^{-\lambda_{mn}^2 t},$$

where $\mu_m = \dfrac{m\pi}{a}$, $\nu_n = \dfrac{n\pi}{b}$, $\lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2}$, *and*

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin(\mu_m x) \sin(\nu_n y)\, dy\, dx.$$

Example

A 2×2 square plate with $c = 1/3$ is heated in such a way that the temperature in the lower half is 50, while the temperature in the upper half is 0. After that, it is insulated laterally, and the temperature at its edges is held at 0. Find an expression that gives the temperature in the plate for $t > 0$.

We must solve the heat problem above with $a = b = 2$ and

$$f(x,y) = \begin{cases} 50 & \text{if } y \le 1, \\ 0 & \text{if } y > 1. \end{cases}$$

The coefficients in the solution are

$$\begin{aligned} A_{mn} &= \frac{4}{2 \cdot 2} \int_0^2 \int_0^2 f(x,y) \sin\left(\frac{m\pi}{2}x\right) \sin\left(\frac{n\pi}{2}y\right) \, dy \, dx \\ &= 50 \int_0^2 \sin\left(\frac{m\pi}{2}x\right) \, dx \int_0^1 \sin\left(\frac{n\pi}{2}y\right) \, dy \end{aligned}$$

$$= 50\left(\frac{2(1+(-1)^{m+1})}{\pi m}\right)\left(\frac{2(1-\cos\frac{n\pi}{2})}{\pi n}\right)$$

$$= \frac{200}{\pi^2}\frac{(1+(-1)^{m+1})(1-\cos\frac{n\pi}{2})}{mn}.$$

Since $\lambda_{mn} = \frac{\pi}{3}\sqrt{\frac{m^2}{4}+\frac{n^2}{4}} = \frac{\pi}{6}\sqrt{m^2+n^2}$, the solution is

$$u(x,y,t) = \frac{200}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{(1+(-1)^{m+1})(1-\cos\frac{n\pi}{2})}{mn}\sin\left(\frac{m\pi}{2}x\right)\right.$$
$$\left.\times\sin\left(\frac{n\pi}{2}y\right)e^{-\pi^2(m^2+n^2)t/36}\right).$$