Introduction to Sturm-Liouville Theory

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Partial Differential Equations
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Sturm-Liouville problems

Definition: A (second order) *Sturm-Liouville (S-L) problem* consists of

- A Sturm-Liouville equation on an interval:

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0, \quad a < x < b,$$

  together with

- *Boundary conditions*, i.e. specified behavior of y at $x = a$ and $x = b$.

Definition: A function $y \not\equiv 0$ that solves an S-L problem is called an *eigenfunction*, and the corresponding value of λ is called its *eigenvalue*.

Examples

- The boundary value problem

$$y'' + \lambda y = 0, \quad y(-p) = y(p), \quad y'(-p) = y'(p),$$

is an S-L problem on the interval $[-p, p]$. We have seen that

$$\text{Eigenvalues: } \lambda = \lambda_n = \left(\frac{n\pi}{p}\right)^2 \quad (n \in \mathbb{N}_0)$$

$$\text{Eigenfunctions: } y = y_n = a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)$$

- The boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = y(L) = 0,$$

is an S-L problem on the interval $[0, L]$. We have seen that

$$\text{Eigenvalues: } \lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad (n \in \mathbb{N})$$

$$\text{Eigenfunctions: } y = y_n = c_n \sin\left(\frac{n\pi x}{L}\right)$$

- The boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = -\kappa y(L),$$

is an S-L problem on the interval $[0, L]$. We have seen that

$$\text{Eigenvalues: } \lambda = \lambda_n = \mu_n^2 \quad \left(\tan \mu_n = -\frac{\mu_n}{\kappa}\right)$$

$$\text{Eigenfunctions: } y = y_n = c_n \sin(\mu_n x)$$

- The boundary value problem

$$x^2 y'' + xy' + (\lambda^2 x^2 - p^2)y = 0, \quad y(0) \text{ finite}, \quad y(a) = 0$$

is an S-L problem on the interval $[0, a]$. We have seen that

$$\text{Eigenvalues: } \lambda^2 = \lambda_n^2 = \left(\frac{\alpha_{pn}}{a}\right)^2 \quad (n \in \mathbb{N})$$

$$\text{Eigenfunctions: } y = y_n = c_n J_p\left(\frac{\alpha_{pn} x}{a}\right)$$

where α_{pn} is the nth positive zero of J_p.

Inner products of eigenfunctions

Suppose (y_j, λ_j), (y_k, λ_k) are eigenfunction/eigenvalue pairs of an S-L problem:

$$\begin{aligned} (py_j')' + (q + \lambda_j r)y_j &= 0, \\ (py_k')' + (q + \lambda_k r)y_k &= 0. \end{aligned}$$

Multiply the first by y_k and the second by y_j, then subtract to get

$$(py_j')'y_k - (py_k')'y_j + (\lambda_j - \lambda_k)y_j y_k r = 0.$$

Moving the λ-terms to one side and "adding zero," we get

$$\begin{aligned} (\lambda_j - \lambda_k)y_j y_k r &= (py_k')'y_j - (py_j')'y_k \\ &= (py_k')'y_j + py_k'y_j' - py_j'y_k' - (py_j')'y_k \\ &= (py_k'y_j - py_j'y_k)' \\ &= \left(p(y_k'y_j - y_j'y_k)\right)'. \end{aligned}$$

If $\lambda_j \neq \lambda_k$, we can divide by $\lambda_j - \lambda_k$ to get

$$y_j y_k r = \frac{\left(p(y_k' y_j - y_j' y_k)\right)'}{\lambda_j - \lambda_k}.$$

Now integrate both sides to obtain:

Proposition

If (y_j, λ_j), (y_k, λ_k) are eigenfunction/eigenvalue pairs for an S-L problem on the interval $[a, b]$ and $\lambda_j \neq \lambda_k$, then their inner product with respect to the weight function $r(x)$ is

$$\begin{aligned}\langle y_j, y_k \rangle &= \int_a^b y_j(x) y_k(x) r(x)\, dx \\ &= \left. \frac{p(x)\left(y_k'(x) y_j(x) - y_j'(x) y_k(x)\right)}{\lambda_j - \lambda_k} \right|_a^b.\end{aligned}$$

Remarks

We will be interested in situations when

$$\langle y_j, y_k \rangle = \left. \frac{p(x)\left(y_k'(x)y_j(x) - y_j'(x)y_k(x)\right)}{\lambda_j - \lambda_k} \right|_a^b = 0,$$

i.e. when eigenfunctions with distinct eigenvalues are *orthogonal*.

Examples include:

- *Periodic* S-L problems, i.e. $p(a) = p(b)$ and

$$y(a) = y(b), \quad y'(a) = y'(b).$$

- S-L problems satisfying $p(a) = 0$ and

$$y(a) \text{ is finite}, \quad y(b) = 0.$$

Example

Use the preceding results to establish the orthogonality of the trigonometric system

$$\left\{1, \cos\left(\frac{\pi x}{p}\right), \cos\left(\frac{2\pi x}{p}\right), \ldots, \sin\left(\frac{\pi x}{p}\right), \sin\left(2\frac{\pi x}{p}\right), \ldots\right\}$$

on the interval $[-p, p]$ *with respect to the weight function* $w(x) = 1$.

The functions $y = \cos\left(\frac{n\pi x}{p}\right)$ and $y = \sin\left(\frac{n\pi x}{p}\right)$ are eigenfunctions of the periodic S-L problem

$$y'' + \lambda y = 0, \quad y(-p) = y(p), \quad y'(p) = y'(-p)$$

with eigenvalue $\lambda = \left(\frac{n\pi}{p}\right)^2$.

Since $r(x) \equiv 1$ in this case, we *automatically* find that

$$\int_{-p}^{p} \cos\left(\frac{m\pi x}{p}\right)\cos\left(\frac{n\pi x}{p}\right)\,dx = \int_{-p}^{p} \cos\left(\frac{m\pi x}{p}\right)\sin\left(\frac{n\pi x}{p}\right)\,dx = \int_{-p}^{p} \sin\left(\frac{m\pi x}{p}\right)\sin\left(\frac{n\pi x}{p}\right)\,dx = 0 \quad \text{for } m \neq n.$$

We must verify orthogonality of $\cos\left(\frac{n\pi x}{p}\right)$ and $\sin\left(\frac{n\pi x}{p}\right)$ manually, since they have *the same eigenvalue*:

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right)\sin\left(\frac{n\pi x}{p}\right)\,dx = \frac{p}{2n\pi}\sin^2\left(\frac{n\pi x}{p}\right)\Bigg|_{-p}^{p} = 0.$$

That covers every case, so we're done.

Example

Use the preceding results to establish the orthogonality of the Bessel function system

$$\left\{ J_p\left(\frac{\alpha_{p1}x}{a}\right), J_p\left(\frac{\alpha_{p2}x}{a}\right), J_p\left(\frac{\alpha_{p3}x}{a}\right), \dots, \right\}$$

on the interval $[0, a]$ *with respect to the weight function* $w(x) = x$.

The function $y = J_p\left(\frac{\alpha_{pn}x}{a}\right)$ is an eigenfunction of the S-L problem

$$x^2y'' + xy' + (\lambda^2x^2 - p^2)y = 0, \quad y(0) \text{ finite}, \quad y(a) = 0$$

with eigenvalue $\lambda^2 = \left(\frac{\alpha_{pn}}{a}\right)^2$. The S-L form of this equation is

$$(xy')' + \left(-\frac{p^2}{x} + \lambda^2 x\right) y = 0,$$

which shows that this problem is of the second type mentioned above. Since $r(x) = x$, we're done.

Regular Sturm-Liouville problems

A *regular* Sturm-Liouville problem on $[a, b]$ has the form

$$
\begin{aligned}
(p(x)y')' + (q(x) + \lambda r(x))y &= 0, \quad a < x < b, \\
c_1 y(a) + c_2 y'(a) &= 0, \qquad (1) \\
d_1 y(b) + d_2 y'(b) &= 0, \qquad (2)
\end{aligned}
$$

where:

- $(c_1, c_2) \neq (0, 0)$ and $(d_1, d_2) \neq (0, 0)$;
- p, p', q and r are continuous on $[a, b]$;
- p and r are positive on $[a, b]$.

Remark: We will focus on the boundary conditions (1) and (2), since they yield orthogonality of eigenfunctions.

Examples

- The boundary value problem

$$y'' + \lambda y = 0, \qquad 0 < x < L,$$
$$y(0) = y(L) = 0,$$

is a regular S-L problem ($p(x) = 1$, $r(x) = 1$, $q(x) = 0$).

- The boundary value problem

$$((x^2 + 1)y')' + (x + \lambda)y = 0, \qquad -1 < x < 1,$$
$$y(-1) = y'(1) = 0,$$

is a regular S-L problem ($p(x) = x^2 + 1$, $q(x) = x$, $r(x) = 1$).

Non-example

Although important, the boundary value problem

$$x^2y'' + xy' + (\lambda^2x^2 - p^2)y = 0, \qquad 0 < x < a,$$
$$y(0) \text{ finite}, \ y(a) = 0,$$

is *not* a regular Sturm-Liouville problem.

In Sturm-Liouville form we had $p(x) = r(x) = x$, $q(x) = -p^2/x$.

- p and r are *not positive* when $x = 0$.
- q is *not continuous* when $x = 0$.
- The boundary condition at $x = 0$ is irregular.

This is an example of a *singular Sturm-Liouville problem.*

Orthogonality in regular S-L problems

Suppose y_j and y_k are eigenfunctions of a regular S-L problem with distinct eigenvalues. The boundary condition at $x = a$ gives

$$\begin{aligned} c_1 y_j(a) + c_2 y_j'(a) &= 0, \\ c_1 y_k(a) + c_2 y_k'(a) &= 0, \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} y_j(a) & y_j'(a) \\ y_k(a) & y_k'(a) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $(c_1, c_2) \neq (0, 0)$ the determinant must be zero:

$$y_j(a) y_k'(a) - y_k(a) y_j'(a) = 0.$$

Likewise, the boundary condition at $x = b$ gives

$$y_j(b) y_k'(b) - y_k(b) y_j'(b) = 0.$$

This means that

$$\langle y_j, y_k \rangle = \left. \frac{p(x)\left(y_k'(x) y_j(x) - y_j'(x) y_k(x)\right)}{\lambda_j - \lambda_k} \right|_a^b = 0.$$

Eigenfunctions and eigenvalues of regular S-L problems

Theorem

The eigenvalues of a regular S-L problem on $[a, b]$ *form an increasing sequence of real numbers*

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

with $\lim_{n\to\infty} \lambda_n = \infty$.

Eigenfunctions corresponding to distinct eigenvalues are orthogonal on $[a, b]$ *with respect to the weight* $r(x)$.

Moreover, the eigenfunction y_n *corresponding to* λ_n *is unique (up to a scalar multiple), and has exactly* $n - 1$ *zeros in the interval* $a < x < b$.

Remarks

- Aside from orthogonality, the proof of this result is beyond the scope of our class.
- Up to the "moreover" statement, this result holds for many irregular S-L problems (as we have seen).
- Orthogonality will help us extract coefficients from *eigen-function expansions* of the form $\sum_{n=1}^{\infty} a_n y_n$.
- The results of S-L theory unify and explain all of the ODE boundary value problems we have encountered throughout the semester!