More on Sturm-Liouville Theory

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Partial Differential Equations
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Recall: Sturm-Liouville problems

Definition: A (second order) *Sturm-Liouville (S-L) problem* consists of

- A Sturm-Liouville equation on an interval:

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0, \quad a < x < b,$$

  together with

- *Boundary conditions*, i.e. specified behavior of y at $x = a$ and $x = b$.

Definition: A function $y \not\equiv 0$ that solves an S-L problem is called an *eigenfunction*, and the corresponding value of λ is called its *eigenvalue*.

Recall: Inner products of eigenfunctions

Proposition

If (y_j, λ_j), (y_k, λ_k) *are eigenfunction/eigenvalue pairs for an S-L problem on the interval* $[a, b]$ *and* $\lambda_j \neq \lambda_k$, *then their inner product with respect to the weight function* $r(x)$ *is*

$$\begin{aligned}\langle y_j, y_k \rangle &= \int_a^b y_j(x) y_k(x) r(x)\, dx \\ &= \left. \frac{p(x)\left(y_k'(x)y_j(x) - y_j'(x)y_k(x)\right)}{\lambda_j - \lambda_k} \right|_a^b.\end{aligned}$$

Remark: Frequently the boundary conditions imply that the RHS is zero, i.e. eigenfunctions with distinct eigenvalues are orthogonal.

Recall: Regularity

An S-L problem is called *regular* if:

- The boundary conditions are of the form

$$\begin{aligned} c_1 y(a) + c_2 y'(a) &= 0, \\ d_1 y(b) + d_2 y'(b) &= 0, \end{aligned}$$

- p, q and r satisfy certain *regularity conditions* on $[a, b]$.

Theorem

The eigenvalues of a regular S-L problem on $[a, b]$ form an increasing sequence of real numbers $\lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty$.

Eigenfunctions corresponding to distinct eigenvalues are orthogonal on $[a, b]$ with respect to the weight $r(x)$.

Moreover, the eigenfunction y_n corresponding to λ_n is unique (up to a scalar multiple), and has exactly $n - 1$ zeros in the interval $a < x < b$.

"Fourier convergence" for S-L problems

The final property of eigenfunctions we will need regards their "completeness."

Theorem

Let $y_1, y_2, y_3, \ldots$ be the eigenfunctions of a (regular) S-L problem on $[a, b]$. If f is piecewise smooth on $[a, b]$, then

$$\frac{f(x+) + f(x-)}{2} = \sum_{n=1}^{\infty} A_n y_n(x),$$

where

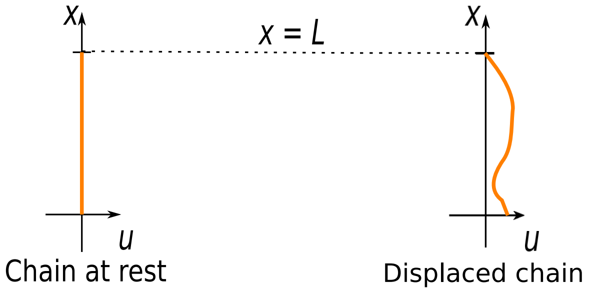
$$A_n = \frac{\langle f, y_n \rangle}{\langle y_n, y_n \rangle} = \frac{\displaystyle\int_a^b f(x) y_n(x) r(x)\, dx}{\displaystyle\int_a^b y_n^2(x) r(x)\, dx}.$$

Remarks

- The series $\sum_{n=1}^{\infty} A_n y_n$ is the *eigenfunction expansion* of f.
- Recall that $f(x) = \frac{f(x+)+f(x-)}{2}$ anywhere f is continuous. So the eigenfunction expansion is equal to f at most points.
- Although we have only stated this result for regular S-L problems, it frequently holds for singular problems as well.
- The "original" Fourier convergence theorem provides an example of this phenomenon (the periodic S-L problem involved in that case is non-regular).

The hanging chain

Consider a chain (or heavy rope, cable, etc.) of length L hanging from a fixed point, subject to only to downward gravitational force.



We place the chain along the (vertical) x-axis, displace the chain from rest, and let

$u(x, t) =$ Horizontal deflection of chain from equilibrium at height x and time t.

Under ideal assumptions (e.g. planar motion, small deflection, no energy loss due to friction or air resistance, etc.) we obtain the boundary value problem

$$\begin{aligned} u_{tt} &= g\left(xu_{xx} + u_x\right), \quad 0 < x < L, \quad t > 0, \\ u(L,t) &= 0, \quad t > 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= v(x), \end{aligned}$$

where

- $f(x)$ is the *initial shape* of the chain,
- $v(x)$ is the *initial (horizontal) velocity* of the chain,
- g is the (constant) acceleration due to gravity.

Separation of variables

Writing $u(x,t) = X(x)T(t)$, separation of variables (and physical considerations) yields

$$\begin{aligned} T'' + \lambda^2 g T &= 0, \quad t > 0, \\ xX'' + X' + \lambda^2 X &= 0, \quad 0 < x < L, \\ X(0) \text{ finite}, \ X(L) &= 0. \end{aligned}$$

The ODE for X can be rewritten as

$$(xX')' + \lambda^2 X = 0,$$

yielding a S-L problem (with $p(x) = x$, $q(x) = 0$, $r(x) = 1$, and parameter λ^2).

Note: Although this S-L problem is non-regular, the stated conditions guarantee orthogonality of eigenfunctions!

To find the eigenfunctions, substitute $s = 2\sqrt{x}$ to get

$$s^2 \frac{d^2X}{ds^2} + s\frac{dX}{ds} + \lambda^2 s^2 X = 0,$$
$$X(0) \text{ finite}, \quad X(2\sqrt{L}) = 0,$$

the parametric Bessel equation of order 0. Therefore

$$\lambda = \lambda_n = \frac{\alpha_n}{2\sqrt{L}}$$
$$X(s) = X_n(s) = J_0\left(\frac{\alpha_n s}{2\sqrt{L}}\right),$$

where α_n is the nth positive zero of J_0. Since $s = 2\sqrt{x}$

$$X(x) = X_n(x) = J_0\left(\alpha_n \sqrt{\frac{x}{L}}\right).$$

The general solution

Since the ODE for T is $T'' + \lambda^2 g T = 0$,

$$T(t) = T_n(t) = A_n \cos\left(\sqrt{g}\lambda_n t\right) + B_n \sin\left(\sqrt{g}\lambda_n t\right)$$
$$= A_n \cos\left(\sqrt{\frac{g}{L}}\frac{\alpha_n t}{2}\right) + B_n \sin\left(\sqrt{\frac{g}{L}}\frac{\alpha_n t}{2}\right),$$

and superposition gives the *general solution*

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$$
$$= \sum_{n=1}^{\infty} J_0\left(\alpha_n \sqrt{\frac{x}{L}}\right)\left(A_n \cos\left(\sqrt{\frac{g}{L}}\frac{\alpha_n t}{2}\right) + B_n \sin\left(\sqrt{\frac{g}{L}}\frac{\alpha_n t}{2}\right)\right).$$

The initial conditions will yield eigenfunction expansions.

The initial shape condition requires that

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \underbrace{J_0\left(\alpha_n\sqrt{\frac{x}{L}}\right)}_{X_n(x)}.$$

According to S-L theory, this means that

$$A_n = \frac{\langle f, X_n\rangle}{\langle X_n, X_n\rangle} = \frac{\int_0^L f(x) J_0\left(\alpha_n\sqrt{\frac{x}{L}}\right)\,dx}{\int_0^L J_0^2\left(\alpha_n\sqrt{\frac{x}{L}}\right)\,dx}$$

$$= \frac{1}{L\,J_1^2(\alpha_n)} \int_0^L f(x) J_0\left(\alpha_n\sqrt{\frac{x}{L}}\right)\,dx.$$

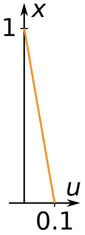
Setting $u_t(x,0) = v(x)$ and using similar reasoning yields

$$B_n = \frac{2}{\alpha_n J_1^2(\alpha_n)\sqrt{gL}} \int_0^L v(x) J_0\left(\alpha_n\sqrt{\frac{x}{L}}\right)\,dx.$$

Example

A 1 meter long chain is suspended at one end. If its lower end is pulled 10cm to the right and then released, find an expression for the shape of the chain at any later time.

We have $L = 1$, $g = 9.8$, $v(x) \equiv 0$ and



$$f(x) = \frac{1-x}{10}.$$

It follows that $B_n = 0$ for all n.

Furthermore

$$A_n = \frac{1}{J_1^2(\alpha_n)} \underbrace{\int_0^1 \frac{1-x}{10} J_0\left(\alpha_n\sqrt{x}\right)\,dx}_{\text{sub. } u=\alpha_n\sqrt{x}}$$

$$= \frac{1}{5\alpha_n^2 J_1^2(\alpha_n)} \int_0^{\alpha_n} \left(1 - \frac{u^2}{\alpha_n^2}\right) J_0(u) u\,du = \frac{1}{5\alpha_n^2 J_1^2(\alpha_n)} 2J_2(\alpha_n)$$

$$= \frac{2}{5\alpha_n^2 J_1^2(\alpha_n)} \left(\frac{2}{\alpha_n} J_1(\alpha_n)\right) = \frac{4}{5\alpha_n^3 J_1(\alpha_n)}.$$

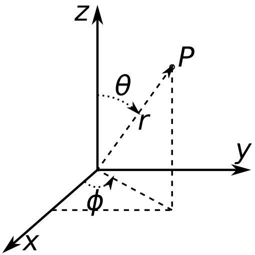
Hence the shape of the chain is given by

$$u(x,t) = \frac{4}{5} \sum_{n=1}^{\infty} \frac{1}{\alpha_n^3 J_1(\alpha_n)} J_0\left(\alpha_n\sqrt{x}\right) \cos\left(\frac{\sqrt{9.8}\alpha_n t}{2}\right).$$

The spherical Dirichlet problem

Spherical coordinates and the spherical Laplacian

Recall: The spherical coordinates of $P = (x, y, z)$ are



$$x = r\sin\theta\cos\phi,$$

$$y = r\sin\theta\sin\phi,$$

$$z = r\cos\theta,$$

$$r^2 = x^2 + y^2 + z^2.$$

One can also show that for a function $u(x, y, z)$:

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(u_{\theta\theta} + \cot(\theta)u_\theta + \csc^2(\theta)u_{\phi\phi}\right).$$

Goal: Determine the steady state temperature in a solid ball of radius a, given a time-independent temperature distribution on its surface.

Set-up: Center the sphere at the origin and work in spherical coordinates.

We assume the surface temperature (and hence the steady state) depends only on θ (latitude), and let

$$u(r,\theta) = \text{temperature in the ball at the (spherical) position } (r,\theta)\ .$$

Since $u_{\phi\phi} \equiv 0$, this yields the boundary value problem

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(u_{\theta\theta} + \cot(\theta)u_\theta\right) = 0, \;\; 0 < \theta < \pi, 0 < r < a,$$
$$u(a,\theta) = f(\theta), \;\; 0 \le \theta \le \pi.$$

Separation of variables

Writing $u(r,\theta) = R(r)\Theta(\theta)$, separation of variables (and physical considerations) yields

$$\begin{aligned}
&r^2R'' + 2rR' - \lambda R = 0, \quad 0 < r < a,\\
&R, R', R'', \dots \text{ finite at } r = 0, a,\\
&\Theta'' + \cot(\theta)\Theta' + \lambda\Theta = 0, \quad 0 < \theta < \pi,\\
&\Theta \text{ finite at } \theta = 0, \pi.
\end{aligned}$$

The ODE for R is an Euler equation with indicial equation

$$\rho^2 + (2-1)\rho - \lambda = \rho^2 + \rho - \lambda = 0 \quad \Rightarrow \quad \rho_\pm = \frac{-1 \pm \sqrt{1+4\lambda}}{2}.$$

Since the solutions are $R = c_1 r^{\rho_+} + c_2 r^{\rho_-}$, finiteness of $R^{(k)}$ implies $c_2 = 0$ and

$$\rho^+ = \frac{-1 + \sqrt{1+4\lambda}}{2} = n \in \mathbb{N}_0 \quad \Rightarrow \quad \lambda = n(n+1).$$

Setting $\lambda = n(n+1)$ and multiplying by $\sin(\theta)$, the Θ problem takes on S-L form:

$$(\sin(\theta)\Theta')' + n(n+1)\sin(\theta)\Theta = 0, \quad 0 < \theta < \pi$$
$$\Theta \text{ finite at } \theta = 0, \pi,$$

with $p(\theta) = r(\theta) = \sin(\theta)$ and $q(\theta) = 0$.

Although this problem is non-regular, the boundary conditions guarantee orthogonality of the eigenfunctions, which have the form

$$\Theta(\theta) = \Theta_n(\theta) = P_n\left(\cos\theta\right),$$

where P_n is the *nth Legendre polynomial* (see 5.5 and 5.6):

$$P_n(x) = \frac{1}{2^n}\sum_{m=0}^{\lfloor n/2 \rfloor}(-1)^m \frac{(2n-2m)!}{m!(n-m)!(n-2m)!}x^{n-2m}.$$

The general solution

Superposition yields the general solution

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n R_n(r)\Theta_n(\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta).$$

The boundary condition requires

$$f(\theta) = u(a,\theta) = \sum_{n=0}^{\infty} A_n a^n \underbrace{P_n(\cos\theta)}_{\Theta_n},$$

and S-L theory therefore implies that

$$A_n = \frac{1}{a^n}\frac{\langle f,\Theta_n\rangle}{\langle \Theta_n,\Theta_n\rangle} = \frac{\int_0^\pi f(\theta)P_n(\cos\theta)\sin\theta\,d\theta}{a^n\int_0^\pi P_n^2(\cos\theta)\sin\theta\,d\theta}$$
$$= \frac{2n+1}{2a^n}\int_0^\pi f(\theta)P_n(\cos\theta)\sin\theta\,d\theta.$$

Example

Find the steady state temperature in a ball of radius 1, if its surface above $z = 1/2$ is kept at 50°, and the remainder is kept at 0°. What is the temperature at the center of the sphere?

Since $a = 1$ and $z = a\cos\theta = \cos\theta$ on the surface of the sphere, we find that the boundary temperature is

$$f(\theta) = \begin{cases} 50 & \text{if } 0 \le \theta < \pi/3, \\ 0 & \text{if } \pi/3 < \theta < \pi. \end{cases}$$

Hence

$$A_n = 25(2n+1) \underbrace{\int_0^{\pi/3} P_n(\cos\theta)\sin(\theta)\,d\theta}_{\text{sub. } x=\cos\theta} = 25(2n+1) \int_{1/2}^{1} P_n(x)\,dx.$$

The coefficient A_n is easy to compute for any given n, e.g.

$$A_3 = 25(2 \cdot 3 + 1) \int_{1/2}^{1} \frac{1}{2} \left(5x^3 - 3x\right) \, dx = \frac{525}{128},$$

although finding a general formula is quite difficult.

According to Maple

n	0	1	2	3	4	5	$\cdots$
A_n	$\frac{25}{2}$	$\frac{225}{8}$	$\frac{375}{16}$	$\frac{525}{128}$	$-\frac{3375}{256}$	$-\frac{15675}{1024}$	$\cdots$

so that

$$u(r,\theta) = \frac{25}{2} + \frac{225}{8} r P_1(\cos\theta) + \frac{375}{16} r^2 P_2(\cos\theta) + \frac{525}{128} r^3 P_3(\cos\theta) + \cdots$$