The Fourier Transform Method

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Recall

The Fourier transform

The *Fourier transform* of a piecewise smooth $f \in L^1(\mathbb{R})$ is

$$\hat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx,$$

and f can be recovered from $\hat{f}$ via the *inverse Fourier transform*

$$f(x) = \mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega.$$

Remarks:

- See Appendix B1 for a table of Fourier transform pairs.
- The Fourier transform can help solve boundary value problems with *unbounded* domains.

Fourier transforms of two-variable functions

If $u(x,t)$ is defined for $-\infty < x < \infty$, we define its *Fourier transform in* x to be

$$\hat{u}(\omega,t) = \mathcal{F}(u(x,t))(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx.$$

Because the Fourier transform treats t as a constant, we have

$$\mathcal{F}\left(\frac{\partial^n u}{\partial x^n}\right) = (i\omega)^n\mathcal{F}(u) = (i\omega)^n\hat{u}$$

and

$$\begin{aligned}\mathcal{F}\left(\frac{\partial^n u}{\partial t^n}\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^n u}{\partial t^n}(x,t)e^{-i\omega x}\,dx \\ &= \frac{\partial^n}{\partial t^n}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,t)e^{-i\omega x}\,dx\right) = \frac{\partial^n}{\partial t^n}\mathcal{F}(u) = \frac{\partial^n\hat{u}}{\partial t^n}.\end{aligned}$$

Example

Solve the 1-D heat equation on an infinite rod,

$$u_t = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x).$$

We take the Fourier transform (in x) on both sides to get

$$\hat{u}_t = c^2(i\omega)^2 \hat{u} = -c^2\omega^2 \hat{u}$$
$$\hat{u}(\omega, 0) = \hat{f}(\omega).$$

Since there is only a t derivative, we solve as though ω were a constant:

$$\widehat{u}(\omega, t) = A(\omega) e^{-c^2\omega^2 t} \quad \Rightarrow \quad \hat{f}(\omega) = \hat{u}(\omega, 0) = A(\omega).$$

To solve for u, we invert the Fourier transform, obtaining

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\omega, t) e^{i\omega x}\, d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-c^2\omega^2 t} e^{i\omega x}\, d\omega. \end{aligned}$$

This expresses the solution in terms of the Fourier transform of the initial temperature distribution $f(x)$.

Remark: By writing $\hat{f}$ as an integral, reversing the order of integration, and using the (inverse) Fourier transform of $e^{-c^2\omega^2 t}$, one can show that in fact

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(s) e^{-(x-s)^2/4c^2 t}\, ds.$$

Example

Solve the boundary value problem

$$u_t = tu_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x),$$

which models the temperature in an infinitely long rod with variable thermal diffusivity.

Taking the Fourier transform (in x) on both sides yields

$$\hat{u}_t = t(i\omega)^2\hat{u} = -t\omega^2\hat{u},$$
$$\hat{u}(\omega,0) = \hat{f}(\omega).$$

The ODE in t is separable, with solution

$$\hat{u}(\omega,t) = A(\omega)e^{-t^2\omega^2/2} \quad \Rightarrow \quad \hat{f}(\omega) = \hat{u}(\omega,0) = A(\omega).$$

As before, Fourier inversion gives

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-t^2\omega^2/2} e^{i\omega x}\, d\omega.$$

In comparison with the preceding example, this decays more rapidly as t increases. This is is physically reasonable, since the thermal diffusivity is increasing with t.

Remark: Notice that this is the solution of the previous example, with $t^2/2$ replacing $c^2 t$. Using the previous remark, this means

$$u(x,t) = \frac{1}{t\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{-(x-s)^2/2t^2}\, ds.$$

Example

Solve the third order mixed derivative boundary value problem

$$u_{tt} = u_{xxt}, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

Taking the Fourier transform (in x) on both sides yields

$$\hat{u}_{tt} = (i\omega)^2 \hat{u}_t = -\omega^2 \hat{u}_t,$$
$$\hat{u}(\omega,0) = \hat{f}(\omega), \quad \hat{u}_t(\omega,0) = \hat{g}(\omega)$$

Solving the ODE it t for $\hat{u}_t$ gives

$$\hat{u}_t(\omega,t) = A(\omega)e^{-\omega^2 t} \quad \Rightarrow \quad \hat{u}(\omega,t) = -\frac{A(\omega)}{\omega^2}e^{-\omega^2 t} + B(\omega)$$
$$= A(\omega)e^{-\omega^2 t} + B(\omega).$$

Imposing the initial conditions we find that

$$\begin{aligned}\hat{f}(\omega) &= \hat{u}(\omega,0) = A(\omega) + B(\omega)\\ \hat{g}(\omega) &= \hat{u}_t(\omega,0) = -\omega^2 A(\omega)\end{aligned} \quad \Rightarrow \quad \begin{aligned} A(\omega) &= \frac{-\hat{g}(\omega)}{\omega^2}\\ B(\omega) &= \hat{f}(\omega) + \frac{\hat{g}(\omega)}{\omega^2}.\end{aligned}$$

Plugging these into $\hat{u}$ and applying Fourier inversion yields

$$\begin{aligned}u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{-\hat{g}(\omega)}{\omega^2} e^{-\omega^2 t} + \hat{f}(\omega) + \frac{\hat{g}(\omega)}{\omega^2} \right) e^{i\omega x}\, d\omega\\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\hat{f}(\omega) + \frac{\hat{g}(\omega)}{\omega^2}(1 - e^{-\omega^2 t}) \right) e^{i\omega x}\, d\omega\\ &= f(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{g}(\omega)}{\omega^2}(1 - e^{-\omega^2 t}) e^{i\omega x}\, d\omega.\end{aligned}$$

Example

Solve the boundary value problem

$$t^2 u_x - u_t = 0, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x),$$

and express the solution explicitly in terms of f.

Taking the Fourier transform (in x) on both sides yields

$$t^2(i\omega)\hat{u} - \hat{u}_t = 0,$$
$$\hat{u}(\omega,0) = \hat{f}(\omega).$$

The ODE in t is separable, with solution

$$\hat{u}(\omega,t) = A(\omega)e^{it^3\omega/3} \quad \Rightarrow \quad \hat{f}(\omega) = \hat{u}(\omega,0) = A(\omega).$$

Using Fourier inversion leads to

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{it^3\omega/3} e^{i\omega x}\, d\omega \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega(x+t^3/3)}\, d\omega \\
&= f\left(x + \frac{t^3}{3}\right).
\end{aligned}
$$

Remark: This particular problem is amenable to the *method of characteristics*, although the Fourier transform method may seem somewhat more straightforward.

Example

Solve the Dirichlet problem in the upper half-plane

$$\nabla^2 u = u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0,$$
$$u(x,0) = f(x),$$

which models the steady state temperature in a semi-infinite plate.

Taking the Fourier transform (in x) on both sides yields

$$(i\omega)^2 \hat{u} + \hat{u}_{yy} = \hat{u}_{yy} - \omega^2 \hat{u} = 0,$$
$$\hat{u}(\omega,0) = \hat{f}(\omega).$$

The ODE in y has characteristic equation

$$r^2 - \omega^2 = 0 \quad \Rightarrow r = \pm\omega \quad \Rightarrow \quad \hat{u}(\omega,y) = A(\omega)e^{\omega y} + B(\omega)e^{-\omega y}.$$

We now require that $\hat{u}(\omega, y)$ remain bounded as $y \to \infty$. Consequently,

$$\left.\begin{array}{lll} \omega > 0 & \Rightarrow & A(\omega) = 0 \\ \omega < 0 & \Rightarrow & B(\omega) = 0 \end{array}\right\} \quad \Rightarrow \quad \hat{u}(\omega, y) = C(\omega) e^{-y|\omega|}$$

$$\Rightarrow \quad \hat{f}(\omega) = \hat{u}(\omega, 0) = C(\omega)$$

Fourier inversion then gives the result

$$\begin{aligned} u(x, y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-y|\omega|} e^{i\omega x}\, d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} f(s) e^{-i\omega(s-x)}\, ds}_{\text{sub. } z=s-x}\ e^{-y|\omega|}\, d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x+z) e^{-i\omega z}\, dz\, e^{-y|\omega|}\, d\omega. \end{aligned}$$

Now reverse the order of integration to obtain

$$\begin{aligned} u(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x+z)\int_{-\infty}^{\infty} e^{-y|\omega|}e^{-i\omega z}\,d\omega\,dz \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x+z)\mathcal{F}_\omega(e^{-y|\omega|})(z)\,dz, \end{aligned}$$

where we have written $\mathcal{F}_\omega$ to indicate the Fourier transform in ω. Recall that

$$\left.\begin{aligned} \mathcal{F}(e^{-|x|}) &= \sqrt{\frac{2}{\pi}}\frac{1}{1+\omega^2} \\ \\ \mathcal{F}(g(ax)) &= \frac{1}{|a|}\hat{g}\left(\frac{\omega}{a}\right) \end{aligned}\right\} \quad\Rightarrow\quad \mathcal{F}(e^{-a|x|}) = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+\omega^2}.$$

Setting $a = y$ and $\omega = z$, it finally follows that

$$u(x,y) = \underbrace{\frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(x+z)}{y^2+z^2}\,dz}_{\text{sub. } s=x+z} = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(s)}{y^2+(x-s)^2}\,ds,$$

which is known as the *Poisson integral formula*.

Remark: The simplification technique above can be generalized by introducing the *convolution* of functions g and h:

$$(g*h)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-t)h(t)\,dt.$$

See Sections 7.2, 7.4 and 7.5.