The Fourier Transform Method

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Partial Differential Equations
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The Fourier transform of a piecewise smooth $f \in L^1(\mathbb{R})$ is

$$\hat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx,$$

and $f$ can be recovered from $\hat{f}$ via the inverse Fourier transform

$$f(x) = \mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, d\omega.$$

Remarks:

- See Appendix B1 for a table of Fourier transform pairs.
- The Fourier transform can help solve boundary value problems with unbounded domains.
Fourier transforms of two-variable functions

If \( u(x, t) \) is defined for \(-\infty < x < \infty\), we define its Fourier transform in \( x \) to be

\[
\hat{u}(\omega, t) = \mathcal{F}(u(x, t))(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{-i\omega x} \, dx.
\]

Because the Fourier transform treats \( t \) as a constant, we have

\[
\mathcal{F} \left( \frac{\partial^n u}{\partial x^n} \right) = (i\omega)^n \mathcal{F}(u) = (i\omega)^n \hat{u}
\]

and

\[
\mathcal{F} \left( \frac{\partial^n u}{\partial t^n} \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n u}{\partial t^n}(x, t) e^{-i\omega x} \, dx
\]

\[
= \frac{\partial^n}{\partial t^n} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{-i\omega x} \, dx \right) = \frac{\partial^n}{\partial t^n} \mathcal{F}(u) = \frac{\partial^n \hat{u}}{\partial t^n}.
\]
Example

Solve the 1-D heat equation on an infinite rod,

$$u_t = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x, 0) = f(x).$$

We take the Fourier transform (in $x$) on both sides to get

$$\hat{u}_t = c^2 (i\omega)^2 \hat{u} = -c^2 \omega^2 \hat{u}$$
$$\hat{u}(\omega, 0) = \hat{f}(\omega).$$

Since there is only a $t$ derivative, we solve as though $\omega$ were a constant:

$$\hat{u}(\omega, t) = A(\omega) e^{-c^2 \omega^2 t} \quad \Rightarrow \quad \hat{f}(\omega) = \hat{u}(\omega, 0) = A(\omega).$$
To solve for $u$, we invert the Fourier transform, obtaining

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\omega, t) e^{i\omega x} \, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-c^2 \omega^2 t} e^{i\omega x} \, d\omega.$$ 

This expresses the solution in terms of the Fourier transform of the initial temperature distribution $f(x)$.

**Remark:** By writing $\hat{f}$ as an integral, reversing the order of integration, and using the (inverse) Fourier transform of $e^{-c^2 \omega^2 t}$, one can show that in fact

$$u(x, t) = \frac{1}{2c \sqrt{\pi t}} \int_{-\infty}^{\infty} f(s) e^{-(x-s)^2/4c^2 t} \, ds.$$
Example

Solve the boundary value problem

\[ u_t = tu_{xx}, \quad -\infty < x < \infty, \quad t > 0, \]
\[ u(x, 0) = f(x), \]

which models the temperature in an infinitely long rod with variable thermal diffusivity.

Taking the Fourier transform (in \( x \)) on both sides yields

\[ \hat{u}_t = t(i\omega)^2 \hat{u} = -t\omega^2 \hat{u}, \]
\[ \hat{u}(\omega, 0) = \hat{f}(\omega). \]

The ODE in \( t \) is separable, with solution

\[ \hat{u}(\omega, t) = A(\omega)e^{-t^2\omega^2/2} \Rightarrow \hat{f}(\omega) = \hat{u}(\omega, 0) = A(\omega). \]
As before, Fourier inversion gives

\[
   u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-t^2\omega^2/2} e^{i\omega x} \, d\omega.
\]

In comparison with the preceding example, this decays more rapidly as \( t \) increases. This is physically reasonable, since the thermal diffusivity is increasing with \( t \).

**Remark:** Notice that this is the solution of the previous example, with \( t^2/2 \) replacing \( c^2t \). Using the previous remark, this means

\[
   u(x, t) = \frac{1}{t\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{-(x-s)^2/2t^2} \, ds.
\]
Example

Solve the third order mixed derivative boundary value problem

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^3 u}{\partial x^3}, & -\infty < x < \infty, \; t > 0, \\
\frac{\partial u}{\partial x}(x, 0) &= f(x), & \frac{\partial u}{\partial t}(x, 0) &= g(x).
\end{align*}
\]

Taking the Fourier transform (in \(x\)) on both sides yields

\[
\begin{align*}
\hat{u}_{tt} &= (i\omega)^2 \hat{u}_t = -\omega^2 \hat{u}_t, \\
\hat{u}(\omega, 0) &= \hat{f}(\omega), & \hat{u}_t(\omega, 0) &= \hat{g}(\omega)
\end{align*}
\]

Solving the ODE it \(t\) for \(\hat{u}_t\) gives

\[
\hat{u}_t(\omega, t) = A(\omega) e^{-\omega^2 t} \Rightarrow \hat{u}(\omega, t) = -\frac{A(\omega)}{\omega^2} e^{-\omega^2 t} + B(\omega)
\]

\[
= A(\omega) e^{-\omega^2 t} + B(\omega).
\]
Imposing the initial conditions we find that

\[
\hat{f}(\omega) = \hat{u}(\omega, 0) = A(\omega) + B(\omega) \\
\hat{g}(\omega) = \hat{u}_t(\omega, 0) = -\omega^2 A(\omega)
\]

\[\Rightarrow\]

\[
A(\omega) = \frac{-\hat{g}(\omega)}{\omega^2} \\
B(\omega) = \hat{f}(\omega) + \frac{\hat{g}(\omega)}{\omega^2}.
\]

Plugging these into \(\hat{u}\) and applying Fourier inversion yields

\[
u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( -\frac{\hat{g}(\omega)}{\omega^2} e^{-\omega^2 t} + \hat{f}(\omega) + \frac{\hat{g}(\omega)}{\omega^2} \right) e^{i\omega x} \, d\omega
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \hat{f}(\omega) + \frac{\hat{g}(\omega)}{\omega^2} (1 - e^{-\omega^2 t}) \right) e^{i\omega x} \, d\omega
\]

\[
= f(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{g}(\omega)}{\omega^2} (1 - e^{-\omega^2 t}) e^{i\omega x} \, d\omega.
\]
Example

Solve the boundary value problem

\[ t^2 u_x - u_t = 0, \quad -\infty < x < \infty, \quad t > 0, \]
\[ u(x, 0) = f(x), \]

and express the solution explicitly in terms of \( f \).

Taking the Fourier transform (in \( x \)) on both sides yields

\[ t^2 (i\omega) \hat{u} - \hat{u}_t = 0, \]
\[ \hat{u}(\omega, 0) = \hat{f}(\omega). \]

The ODE in \( t \) is separable, with solution

\[ \hat{u}(\omega, t) = A(\omega) e^{it^3 \omega/3} \quad \Rightarrow \quad \hat{f}(\omega) = \hat{u}(\omega, 0) = A(\omega). \]
Using Fourier inversion leads to

\[
    u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{it^3\omega/3}e^{i\omega x} \, d\omega
    = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega(x+t^3/3)} \, d\omega
    = f \left( x + \frac{t^3}{3} \right).
\]

**Remark:** This particular problem is amenable to the *method of characteristics*, although the Fourier transform method may seem somewhat more straightforward.
Example

Solve the Dirichlet problem in the upper half-plane

\[ \nabla^2 u = u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0, \]
\[ u(x, 0) = f(x), \]

which models the steady state temperature in a semi-infinite plate.

Taking the Fourier transform (in \( x \)) on both sides yields

\[ (i\omega)^2 \hat{u} + \hat{u}_{yy} = \hat{u}_{yy} - \omega^2 \hat{u} = 0, \]
\[ \hat{u}(\omega, 0) = \hat{f}(\omega). \]

The ODE in \( y \) has characteristic equation

\[ r^2 - \omega^2 = 0 \quad \Rightarrow \quad r = \pm \omega \quad \Rightarrow \quad \hat{u}(\omega, y) = A(\omega)e^{\omega y} + B(\omega)e^{-\omega y}. \]
We now require that $\hat{u}(\omega, y)$ remain bounded as $y \to \infty$. Consequently,

\[
\begin{align*}
\omega > 0 & \quad \Rightarrow \quad A(\omega) = 0 \quad \Rightarrow \quad \hat{u}(\omega, y) = C(\omega) e^{-y|\omega|} \\
\omega < 0 & \quad \Rightarrow \quad B(\omega) = 0 \quad \Rightarrow \quad \hat{f}(\omega) = \hat{u}(\omega, 0) = C(\omega)
\end{align*}
\]

Fourier inversion then gives the result

\[
u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-y|\omega|} e^{i\omega x} \, d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underbrace{f(s) e^{-i\omega(s-x)}}_{\text{sub. } z=s-x} \, ds \, e^{-y|\omega|} \, d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x + z) e^{-i\omega z} \, dz \, e^{-y|\omega|} \, d\omega.
\]
Now reverse the order of integration to obtain

\[ u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x + z) \int_{-\infty}^{\infty} e^{-y|\omega|} e^{-i\omega z} \, d\omega \, dz \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x + z) \mathcal{F}_\omega(e^{-y|\omega|})(z) \, dz, \]

where we have written \( \mathcal{F}_\omega \) to indicate the Fourier transform in \( \omega \). Recall that

\[
\begin{align*}
\mathcal{F}(e^{-|x|}) &= \sqrt{\frac{2}{\pi}} \frac{1}{1 + \omega^2} \\
\mathcal{F}(g(ax)) &= \frac{1}{|a|} \hat{g} \left( \frac{\omega}{a} \right)
\end{align*}
\]

\[
\Rightarrow \quad \mathcal{F}(e^{-a|x|}) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + \omega^2}.
\]
Setting $a = y$ and $\omega = z$, it finally follows that

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x + z)}{y^2 + z^2} \, dz = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{y^2 + (x - s)^2} \, ds,$$

which is known as the *Poisson integral formula*.

**Remark:** The simplification technique above can be generalized by introducing the *convolution* of functions $g$ and $h$:

$$(g \ast h)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x - t)h(t) \, dt.$$

See Sections 7.2, 7.4 and 7.5.