The Wave Equation on a Disk

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Partial Differential Equations
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The vibrating circular membrane

Goal: Model the motion of an elastic membrane stretched over a circular frame of radius a.

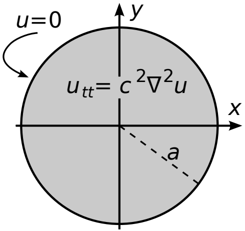
Set-up: Center the membrane at the origin in the xy-plane and let

$$u(r,\theta,t) = \begin{array}{l}\text{deflection of membrane from equilibrium at} \\ \text{polar position } (r,\theta) \text{ and time } t.\end{array}$$

Under ideal assumptions:

$$u_{tt} = c^2\nabla^2 u = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right),$$
$$0 < r < a, \quad 0 < \theta < 2\pi, \quad t > 0,$$

$$u(a,\theta,t) = 0, \quad 0 \le \theta \le 2\pi, \quad t > 0.$$



Separation of variables

Setting $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$ leads to the separated boundary value problems

$$\begin{aligned}
&r^2R'' + rR' + \left(\lambda^2r^2 - \mu^2\right)R = 0, \quad R(0) \text{ finite}, \quad R(a) = 0,\\
&\Theta'' + \mu^2\Theta = 0, \quad \Theta \ \ 2\pi\text{-periodic},\\
&T'' + c^2\lambda^2T = 0.
\end{aligned}$$

We have already seen that the second yields

$$\Theta(\theta) = \Theta_m(\theta) = A\cos(m\theta) + B\sin(m\theta), \quad \mu = m \in \mathbb{N}_0.$$

For each such μ, we have also seen that the solution to the first is

$$R(r) = J_m(\lambda r),$$

where J_m is the Bessel function of the first kind of order m.

We still have one more boundary condition:

$$R(a) = 0 \quad \Rightarrow \quad J_m(\lambda a) = 0 \quad \Rightarrow \lambda a = \alpha_{mn},$$

where α_{mn} is the nth positive zero of J_m. This means that

$$\lambda = \lambda_{mn} = \frac{\alpha_{mn}}{a},$$

and hence $R(r) = R_{mn}(r) = J_m\left(\lambda_{mn} r\right)$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}$.

Returning to T, we finally find that

$$T(t) = T_{mn}(t) = C \cos\left(c\lambda_{mn} t\right) + D \sin\left(c\lambda_{mn} t\right).$$

Normal modes

Multiplying our results together gives the separated solutions

$$J_m(\lambda_{mn} r)\left(A\cos(m\theta) + B\sin(m\theta)\right)\left(C\cos(c\lambda_{mn} t) + D\sin(c\lambda_{mn} t)\right).$$

For convenience we split these in two and write

$$\begin{aligned} u_{mn}(r,\theta,t) &= J_m(\lambda_{mn} r)\left(a_{mn}\cos(m\theta) + b_{mn}\sin(m\theta)\right)\cos(c\lambda_{mn} t), \\ u_{mn}^*(r,\theta,t) &= J_m(\lambda_{mn} r)\left(a_{mn}^*\cos(m\theta) + b_{mn}^*\sin(m\theta)\right)\sin(c\lambda_{mn} t). \end{aligned}$$

Note that, up to scaling, rotation and a phase shift in time, these have the form

$$u(r,\theta,t) = J_m(\lambda_{mn} r)\,\cos(m\theta)\,\cos(c\lambda_{mn} t).$$

Remarks

Let's compare with the normal modes of the rectangular membrane problem:

$$u(x, y, t) = \sin(\mu_m x) \sin(\nu_n y) \cos(\lambda_{mn} t)$$

- The functions $R_{mn}(r)$ are the polar analogs of

$$X_m(x) = \sin(\mu_m x), \quad Y_n(x) = \sin(\nu_n y).$$

- The numbers $\lambda_{mn} = \frac{\alpha_{mn}}{a}$ are analogous to $\mu_m = \frac{m\pi}{a}$ and $\nu_n = \frac{n\pi}{b}$.

Moral: We have (essentially) replaced sine by J_m and the zeros of sine by those of J_m.

Initial conditions and superposition

In order to completely determine the shape of the membrane at any time we must specify the *initial conditions*

$$u(r,\theta,0) = f(r,\theta),\ \ 0 \le r \le a,\ \ 0 \le \theta \le 2\pi\ \ \text{(shape)},$$
$$u_t(r,\theta,0) = g(r,\theta),\ \ 0 \le r \le a,\ \ 0 \le \theta \le 2\pi\ \ \text{(velocity)}.$$

In order to meet these conditions we use superposition to build the *general solution*

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \underbrace{J_m(\lambda_{mn} r)\left(a_{mn}\cos(m\theta) + b_{mn}\sin(m\theta)\right)\cos(c\lambda_{mn} t)}_{u_{mn}(r,\theta,t)}$$
$$+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \underbrace{J_m(\lambda_{mn} r)\left(a^*_{mn}\cos(m\theta) + b^*_{mn}\sin(m\theta)\right)\sin(c\lambda_{mn} t)}_{u^*_{mn}(r,\theta,t)}.$$

Othogonality of Bessel functions

We will see later that the functions $R_{mn}(r) = J_m(\lambda_{mn} r)$ are orthogonal relative to the *weighted inner product*

$$\langle f, g \rangle = \int_0^a f(r) g(r) r \, dr.$$

That is,

$$\langle R_{mn}, R_{mk} \rangle = \int_0^a J_m(\lambda_{mn} r) \, J_m(\lambda_{mk} r) \, r \, dr = 0 \quad \text{if } n \neq k.$$

In addition, it can also be shown that

$$\langle R_{mn}, R_{mn} \rangle = \int_0^a J_m^2(\lambda_{mn} r) r \, dr = \frac{a^2}{2} J_{m+1}^2(\alpha_{mn}).$$

Using the orthogonality relations for Bessel and trigonometric functions, one obtains:

Theorem

The functions

$$\phi_{mn}(r,\theta) = J_m(\lambda_{mn} r)\cos(m\theta),$$
$$\psi_{mn}(r,\theta) = J_m(\lambda_{mn} r)\sin(m\theta),$$

($m \in \mathbb{N}_0$, $n \in \mathbb{N}$) form a (complete) orthogonal set of functions relative to the inner product

$$\langle f, g\rangle = \int_0^{2\pi}\int_0^a f(r,\theta)g(r,\theta)r\,dr\,d\theta.$$

That is, $\langle \phi_{mn}, \phi_{jk}\rangle = \langle \psi_{mn}, \psi_{jk}\rangle = 0$ for $(m,n) \neq (j,k)$ and $\langle \phi_{mn}, \psi_{jk}\rangle = 0$ for all (m,n) and (j,k).

Imposing the initial conditions

Setting $t = 0$ in the general solution gives

$$f(r,\theta) = u(r,\theta,0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a_{mn}\cos(m\theta) + b_{mn}\sin(m\theta)\right)$$

$$= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \left(a_{mn}\phi_{mn}(r,\theta) + b_{mn}\psi_{mn}(r,\theta)\right),$$

$$g(r,\theta) = u_t(r,\theta,0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c\lambda_{mn} J_m(\lambda_{mn} r)\left(a^*_{mn}\cos(m\theta) + b^*_{mn}\sin(m\theta)\right)$$

$$= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \left(c\lambda_{mn} a^*_{mn}\phi_{mn}(r,\theta) + c\lambda_{mn} b^*_{mn}\psi_{mn}(r,\theta)\right),$$

which are called *Fourier-Bessel expansions*.

Integral formulae for a_{mn} and b_{mn}

Using orthogonality, the usual argument gives

$$a_{mn} = \frac{\langle f, \phi_{mn} \rangle}{\langle \phi_{mn}, \phi_{mn} \rangle} = \frac{\displaystyle\int_0^{2\pi} \int_0^a f(r,\theta) J_m(\lambda_{mn} r) \cos(m\theta)\, r\, dr\, d\theta}{\displaystyle\int_0^{2\pi} \int_0^a J_m^2(\lambda_{mn} r)\, \cos^2(m\theta)\, r\, dr\, d\theta}.$$

for $m \geq 0$, $n \geq 1$. Using the complementary orthogonality relation, the integral in the denominator is equal to

$$\int_0^{2\pi} \cos^2(m\theta)\, d\theta \int_0^a J_m^2(\lambda_{mn} r)\, r\, dr = \begin{cases} \pi a^2 J_1^2(\alpha_{0n}) & \text{if } m = 0, \\ \dfrac{\pi a^2}{2} J_{m+1}^2(\alpha_{mn}) & \text{if } m \geq 1. \end{cases}$$

We finally find that

$$a_{0n} = \frac{1}{\pi a^2 J_1^2(\alpha_{0n})} \int_0^{2\pi} \int_0^a f(r,\theta)\, J_0(\lambda_{0n} r)\, r\, dr\, d\theta,$$

$$a_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \int_0^a f(r,\theta)\, J_m(\lambda_{mn} r)\, \cos(m\theta)\, r\, dr\, d\theta,$$

and likewise (using ψ_{mn} instead)

$$b_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \int_0^a f(r,\theta)\, J_m(\lambda_{mn} r)\, \sin(m\theta)\, r\, dr\, d\theta,$$

for $m, n \in \mathbb{N}$.

Integral formulae for a^*_{mn} and b^*_{mn}

The same reasoning using the Fourier-Bessel expansion of $g(x, y)$ yields

$$\begin{aligned}
a^*_{0n} &= \frac{1}{\pi c \alpha_{0n} a J_1^2(\alpha_{0n})} \int_0^{2\pi} \int_0^a g(r,\theta)\, J_0(\lambda_{0n} r)\, r\, dr\, d\theta, \\
a^*_{mn} &= \frac{2}{\pi c \alpha_{mn} a J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \int_0^a g(r,\theta)\, J_m(\lambda_{mn} r)\, \cos(m\theta)\, r\, dr\, d\theta, \\
b^*_{mn} &= \frac{2}{\pi c \alpha_{mn} a J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi} \int_0^a g(r,\theta)\, J_m(\lambda_{mn} r)\, \sin(m\theta)\, r\, dr\, d\theta,
\end{aligned}$$

for $m, n \in \mathbb{N}$.

This (almost) completes the statement of the general solution to the vibrating circular membrane problem!

Remark

Since $\cos 0 = 1$ and $\sin 0 = 0$ we have

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a_{mn}\cos(m\theta) + b_{mn}\sin(m\theta)\right)\cos(c\lambda_{mn} t)$$

$$= \underbrace{\sum_{n=1}^{\infty} a_{0n} J_0(\lambda_{0n} r)\cos(c\lambda_{0n} t)}_{m=0} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\text{as above})$$

- Note that there are really *no* b_{0n} coefficients.
- This is the "true form" of the first series in the solution.

Analogous comments hold for the second series.

Remark

If $f(r,\theta) = f(r)$ (i.e. f is *radially symmetric*), then for $m \neq 0$

$$\begin{aligned} a_{mn} &= (\cdots) \int_0^{2\pi} \int_0^a f(r)\, J_m(\lambda_{mn} r)\, \cos(m\theta)\, r\, dr\, d\theta \\ &= (\cdots) \int_0^a \cdots\, dr\, \underbrace{\int_0^{2\pi} \cos(m\theta)\, d\theta}_{0} = 0, \end{aligned}$$

and $b_{mn} = 0$, too. That is, there are *only* a_{0n} terms.

Likewise, if g is radially symmetric, then for $m \neq 0$

$$a^*_{mn} = b^*_{mn} = 0,$$

and there are *only* a^*_{0n} terms.

Example

Solve the vibrating membrane problem with $a = c = 1$ and initial conditions

$$f(r,\theta) = 1 - r^4, \qquad g(r,\theta) = 0.$$

Because $g(r,\theta) = 0$, we immediately find that $a^*_{mn} = b^*_{mn} = 0$ for all m and n.

Because f is radially symmetric, we only need to compute a_{0n}. Since $a = 1$, $\lambda_{mn} = \alpha_{mn}$, so

$$\begin{aligned} a_{0n} &= \frac{1}{\pi J_1^2(\alpha_{0n})} \int_0^{2\pi} \int_0^1 f(r) J_0(\alpha_{0n} r) r \, dr d\theta \\ &= \frac{2}{J_1^2(\alpha_{0n})} \underbrace{\int_0^1 (1 - r^4) J_0(\alpha_{0n} r) r \, dr}_{\text{substitute } x = \alpha_{0n} r} \end{aligned}$$

$$= \frac{2}{\alpha_{0n}^2 J_1^2(\alpha_{0n})} \int_0^{\alpha_{0n}} \left(1 - \frac{x^4}{\alpha_{0n}^4}\right) J_0(x) x \, dx$$

$$= \frac{2}{\alpha_{0n}^2 J_1^2(\alpha_{0n})} \left(\underbrace{\int_0^{\alpha_{0n}} x J_0(x) \, dx}_{A} - \frac{1}{\alpha_{0n}^4} \underbrace{\int_0^{\alpha_{0n}} x^5 J_0(x) \, dx}_{B} \right).$$

According to earlier results

$$A = \int_0^{\alpha_{0n}} x J_0(x) \, dx = x J_1(x) \Big|_0^{\alpha_{0n}} = \alpha_{0n} J_1(\alpha_{0n}),$$

$$B = \int_0^{\alpha_{0n}} x^5 J_0(x) \, dx = x^5 J_1(x) - 4x^4 J_2(x) + 8x^3 J_3(x) \Big|_0^{\alpha_{0n}}$$

$$= \alpha_{0n}^5 J_1(\alpha_{0n}) - 4\alpha_{0n}^4 J_2(\alpha_{0n}) + 8\alpha_{0n}^3 J_3(\alpha_{0n}).$$

It follows that

$$a_{0n} = \frac{2}{\alpha_{0n}^2 J_1^2(\alpha_{0n})}\left(A - \frac{1}{\alpha_{0n}^4}B\right) = \frac{8\left(\alpha_{0n}J_2(\alpha_{0n}) - 2J_3(\alpha_{0n})\right)}{\alpha_{0n}^3 J_1^2(\alpha_{0n})},$$

so that finally

$$u(r,\theta,t) = \sum_{n=1}^{\infty} \frac{8\left(\alpha_{0n}J_2(\alpha_{0n}) - 2J_3(\alpha_{0n})\right)}{\alpha_{0n}^3 J_1^2(\alpha_{0n})} J_0(\alpha_{0n}r)\cos(\alpha_{0n}t).$$

Remark: This solution can easily be implemented in Maple, since the command

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BesselJZeros(m,n)
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will compute α_{mn} numerically.