More Examples of the Circular Membrane

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Recall

The shape of a vibrating circular membrane of radius a is given by

$$u(r,\theta,t) = \sum_{n=1}^{\infty} a_{0n} J_0(\lambda_{0n} r)\cos(c\lambda_{0n} t)$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a_{mn}\cos(m\theta) + b_{mn}\sin(m\theta)\right)\cos(c\lambda_{mn} t)$$
$$+ \sum_{n=1}^{\infty} a_{0n}^{*} J_0(\lambda_{0n} r)\sin(c\lambda_{0n} t)$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)\left(a_{mn}^{*}\cos(m\theta) + b_{mn}^{*}\sin(m\theta)\right)\sin(c\lambda_{mn} t)$$

where...

Recall (continued)

- $\lambda_{mn} = \dfrac{\alpha_{mn}}{a}$ and α_{mn} is the nth positive zero of $J_m(x)$.
- a_{mn}, b_{mn} (resp. a^*_{mn}, b^*_{mn}) are given by integrals involving the initial shape (resp. initial velocity), e.g. when $m \neq 0$

$$a_{mn} = \frac{\overbrace{2}^{1 \text{ when } m=0}}{\underbrace{\pi a^2}_{\pi c \alpha_{mn} a \text{ for } *} J^2_{m+1}(\alpha_{mn})} \int_0^{2\pi} \int_0^a \overbrace{f(r,\theta)}^{g(r,\theta) \text{ for } *} \underbrace{\cos(m\theta)}_{\text{sine for } b_{mn}} J_m(\lambda_{mn} r) r \, dr \, d\theta.$$

- $a_{mn} = b_{mn} = 0$ (resp. $a^*_{mn} = b^*_{mn} = 0$) for $m \neq 0$ when f (resp. g) is *radially symmetric*, i.e. $f(r,\theta) = f(r)$.

A non-symmetric example

Example

Solve the vibrating membrane problem with $a = c = 1$ and initial conditions

$$f(r,\theta) = r(1-r^4)\cos\theta, \qquad g(r,\theta) = 0.$$

Since $g \equiv 0$, $a^*_{mn} = b^*_{mn} = 0$ for all m, n. We also have

$$\begin{aligned} b_{mn} &= \frac{2}{\pi J^2_{m+1}(\alpha_{mn})} \int_0^{2\pi}\int_0^1 r(1-r^4)\cos\theta J_m(\alpha_{mn} r)\sin(m\theta) r\,dr d\theta \\ &= \frac{2}{\pi J^2_{m+1}(\alpha_{mn})} \underbrace{\int_0^{2\pi} \cos\theta \sin(m\theta)\,d\theta}_{0} \int_0^1 r(1-r^4) J_m(\alpha_{mn} r) r\,dr \\ &= 0 \qquad \text{for all } m, n. \end{aligned}$$

Additionally,

$$\begin{aligned} a_{0n} &= \frac{1}{\pi J_1^2(\alpha_{0n})} \int_0^{2\pi}\int_0^1 r(1-r^4)\,\cos\theta J_0(\alpha_{0n} r) r\, dr d\theta \\ &= \frac{1}{\pi J_1^2(\alpha_{0n})} \underbrace{\int_0^{2\pi} \cos\theta\, d\theta}_{0} \int_0^1 r(1-r^4) J_0(\alpha_{0n} r) r\, dr \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} a_{mn} &= \frac{2}{\pi J_{m+1}^2(\alpha_{mn})} \int_0^{2\pi}\int_0^1 r(1-r^4)\,\cos\theta J_m(\alpha_{mn} r) \cos(m\theta) r\, dr d\theta \\ &= \frac{2}{\pi J_{m+1}^2(\alpha_{mn})} \underbrace{\int_0^{2\pi} \cos\theta\, \cos(m\theta)\, d\theta}_{A} \int_0^1 r(1-r^4) J_m(\alpha_{mn} r) r\, dr. \end{aligned}$$

The integral A is zero unless $m = 1$, in which case it's equal to π. In this case

$$a_{1n} = \frac{2}{J_2^2(\alpha_{1n})} \int_0^1 r(1 - r^4) J_1(\alpha_{1n} r) r \, dr$$
$$= \frac{2}{J_2^2(\alpha_{1n})} \left(\int_0^1 r^2 J_1(\alpha_{1n} r) \, dr - \int_0^1 r^6 J_1(\alpha_{1n} r) \, dr \right).$$

Substituting $x = \alpha_{1n} r$ and proceeding as before one can show

$$\int_0^1 r^2 J_1(\alpha_{1n} r) \, dr = \frac{J_2(\alpha_{1n})}{\alpha_{1n}},$$

$$\int_0^1 r^6 J_1(\alpha_{1n} r) \, dr = \frac{J_2(\alpha_{1n})}{\alpha_{1n}} - \frac{4 J_3(\alpha_{1n})}{\alpha_{1n}^2} + \frac{8 J_4(\alpha_{1n})}{\alpha_{1n}^3}.$$

Assembling these formulae gives

$$a_{1n} = \frac{2}{J_2^2(\alpha_{1n})}\left(\frac{4J_3(\alpha_{1n})}{\alpha_{1n}^2} - \frac{8J_4(\alpha_{1n})}{\alpha_{1n}^3}\right) = \frac{8\left(\alpha_{1n}J_3(\alpha_{1n}) - 2J_4(\alpha_{1n})\right)}{\alpha_{1n}^3 J_2^2(\alpha_{1n})}.$$

Since all the other coefficients are zero,

$$u(r,\theta,t) = \cos\theta \sum_{n=1}^{\infty} \frac{8\left(\alpha_{1n}J_3(\alpha_{1n}) - 2J_4(\alpha_{1n})\right)}{\alpha_{1n}^3 J_2^2(\alpha_{1n})} J_1(\alpha_{1n} r)\cos(\alpha_{1n} t).$$

Remark: In general, one should **not** expect the solution to reduce to a single series.

A “complicated” example

Example

Solve the vibrating membrane problem with $a = 2$, $c = 1$ and initial conditions

$$f(r,\theta) = 0, \qquad g(r,\theta) = r^2(2-r)\sin^8\left(\frac{\theta}{2}\right).$$

Since $f \equiv 0$, $a_{mn} = 0$, $b_{mn} = 0$. We also have

$$b^*_{mn} = (\cdots)\int_0^2 (\cdots)\,dr \int_0^{2\pi} \underbrace{\sin^8\left(\frac{\theta}{2}\right)\sin(m\theta)}_{\text{odd, } 2\pi\text{-periodic}}\,d\theta = 0,$$

$$a^*_{0n} = \frac{1}{\pi\alpha_{0n}2J_1^2(\alpha_{0n})} \underbrace{\int_0^{2\pi} \sin^8\left(\frac{\theta}{2}\right)\,d\theta}_{35\pi/64 \text{ (Maple)}} \underbrace{\int_0^2 r^2(2-r)J_0(\lambda_{0n}r)r\,dr}_{?},$$

and

$$a^*_{mn} = \frac{2}{\pi \alpha_{mn} 2 J^2_{m+1}(\alpha_{mn})} \underbrace{\int_0^{2\pi} \sin^8\left(\frac{\theta}{2}\right) \cos(m\theta)\, d\theta}_{0 \text{ if } m \geq 5 \text{ (Maple)}} \underbrace{\int_0^2 r^2(2-r) J_m(\lambda_{mn} r) r\, dr}_{?}.$$

The solution therefore can be written

$$u(r,\theta,t) = \sum_{m=0}^{4} \sum_{n=1}^{\infty} a^*_{mn} J_m(\lambda_{mn} r) \cos(m\theta) \sin(\lambda_{mn} t),$$

although the (?) integrals are not amenable to evaluation by hand.