Weighted Inner Products and Sturm-Liouville Equations

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Inner products with weight functions

If $w(x) \geq 0$ for $x \in [a, b]$ we define the *inner product on* $[a, b]$ *with respect to the weight* w to be

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)\,dx.$$

We say f and g are *orthogonal on* $[a, b]$ *with respect to the weight* w if

$$\langle f, g \rangle = 0.$$

Remarks:

- The inner product and orthogonality depend on the choice of a, b and w.
- When $w(x) \equiv 1$, these definitions reduce to the "ordinary" ones.

Examples

1. The functions $f_n(x) = \sin(nx)$ $(n = 1, 2, \ldots)$ are pairwise orthogonal on $[0, \pi]$ with respect to the weight function $w(x) \equiv 1$.

2. For a fixed $p \geq 0$, the functions $f_n(x) = J_p(\alpha_{pn}x/a)$ are pairwise orthogonal on $[0, a]$ with respect to the weight function $w(x) = x$.

3. The functions

$$f_0(x) = 1, \quad f_1(x) = 2x, \quad f_2(x) = 4x^2 - 1, \quad f_3(x) = 8x^3 - 4x,$$
$$f_4(x) = 16x^4 - 12x^2 + 1, \qquad f_5(x) = 32x^5 - 32x^3 + 6x$$

are pairwise orthogonal on $[-1, 1]$ with respect to the weight function $w(x) = \sqrt{1 - x^2}$.

Series expansions

Weighted inner products have exactly the same algebraic properties as the "ordinary" inner product. In particular, we can deduce the following fact in the usual way.

Theorem

Suppose that $\{f_1, f_2, f_3, \ldots\}$ is an orthogonal set of functions on $[a, b]$ with respect to the weight function w. If

$$f(x) = \sum_n a_n f_n(x), \quad \textit{(generalized Fourier series)}$$

then the coefficients a_n are given by

$$a_n = \frac{\langle f, f_n \rangle}{\langle f_n, f_n \rangle} = \frac{\int_a^b f(x) f_n(x) w(x)\, dx}{\int_a^b f_n^2(x) w(x)\, dx}.$$

Example

Express the polynomial $f(x) = x^5 - 1$ as a linear combination of the polynomials f_0, f_1, f_2, f_3, f_4 and f_5 from above.

Since the relevant weight function is $w(x) = \sqrt{1-x^2}$ on $[-1,1]$, we compute

$$a_0 = \frac{\langle x^5-1, 1\rangle}{\langle 1,1\rangle} = \frac{\int_{-1}^{1}(x^5-1)\sqrt{1-x^2}\,dx}{\int_{-1}^{1}\sqrt{1-x^2}\,dx} = \frac{-\pi/2}{\pi/2} = -1,$$

$$a_1 = \frac{\langle x^5-1, 2x\rangle}{\langle 2x,2x\rangle} = \frac{\int_{-1}^{1}2x(x^5-1)\sqrt{1-x^2}\,dx}{\int_{-1}^{1}4x^2\sqrt{1-x^2}\,dx} = \frac{5\pi/64}{\pi/2} = \frac{5}{32},$$

$$a_2 = \frac{\langle x^5-1, 4x^2-1\rangle}{\langle 4x^2-1,4x^2-1\rangle} = \frac{\int_{-1}^{1}(4x^2-1)(x^5-1)\sqrt{1-x^2}\,dx}{\int_{-1}^{1}(4x^2-1)^2\sqrt{1-x^2}\,dx}$$

$$= \frac{0}{\pi/2} = 0.$$

In a similar way we find that

$$a_3 = \frac{1}{8}, \quad a_4 = 0, \quad \text{and } a_5 = \frac{1}{32}.$$

Hence

$$x^5 - 1 = -f_0(x) + \frac{5}{32} f_1(x) + \frac{1}{8} f_3(x) + \frac{1}{32} f_5(x),$$

which is easily checked directly.

Remark: In this case, inner products are *not* the most efficient way to find a_0, a_1, etc.

Sturm-Liouville equations

A (second order) *Sturm-Liouville equation* has the form

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0.$$

where p, q and r are specific functions, and λ is a parameter.

Remarks:

- Because λ is a parameter, it is frequently replaced by other variables or expressions.
- Many "familiar" ODEs that occur during separation of variables can be put in Sturm-Liouville form.

Example

Show that $y'' + \lambda y = 0$ is a Sturm-Liouville equation.

Take $p(x) = r(x) = 1$ and $q(x) = 0$.

Example

Put the parametric Bessel equation

$$x^2y'' + xy' + (\lambda^2x^2 - m^2)y = 0$$

in Sturm-Liouville form.

First we divide by x to get

$$\underbrace{xy'' + y'}_{(xy')'} + \left(\lambda^2 x - \frac{m^2}{x}\right) y = 0.$$

This is in Sturm-Liouville form with $p(x) = x$, $q(x) = -\frac{m^2}{x}$, $r(x) = x$, and parameter λ^2.

Example

Put Legendre's differential equation

$$y'' - \frac{2x}{1-x^2}y' + \frac{\mu}{1-x^2}y = 0$$

in Sturm-Liouville form.

First we multiply by $1 - x^2$ to get

$$\underbrace{(1-x^2)y'' - 2xy'}_{((1-x^2)y')'} + \mu y = 0.$$

This is in Sturm-Liouville form with

$$p(x) = 1 - x^2, \quad q(x) = 0, \quad r(x) = 1,$$

provided we write the parameter as μ.

Example

Put Chebyshev's differential equation

$$(1-x^2)y'' - xy' + n^2 y = 0$$

in Sturm-Liouville form.

First we divide by $\sqrt{1-x^2}$ to get

$$\underbrace{\sqrt{1-x^2}\,y'' - \frac{x}{\sqrt{1-x^2}}\,y'}_{(\sqrt{1-x^2}\,y')'} + \frac{n^2}{\sqrt{1-x^2}}\,y = 0.$$

This is in Sturm-Liouville form with

$$p(x) = \sqrt{1-x^2}, \quad q(x) = 0, \quad r(x) = \frac{1}{\sqrt{1-x^2}},$$

provided we write the parameter as n^2.